

Pole Placement Using Optimal Regulators

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A method for constructing a linear quadratic regulator with prescribed closed-loop poles is presented. The design method employs successive shifting of either a single real pole or a pair of complex conjugate poles at a time. This imposes a certain limitation to the location of the closed-loop poles to be specified, and the region of assignable poles is clarified. The effectiveness of the proposed method is illustrated by a numerical example.

Keywords: pole placement, optimal regulator, weighting matrix, successive pole shifting

1. Introduction

Linear quadratic regulators have been considered as a standard means for the design of control systems. The stability of the closed-loop system is guaranteed with a certain gain margin for arbitrary weighting matrices of the performance criteria, provided the assumptions regarding positive definiteness, controllability and observability are satisfied.

However, the relationship between the weighting matrices and the closed-loop poles is not straightforward except for special values of the weighting matrices⁽¹⁾⁻⁽³⁾, and trial and error for an appropriate selection of the weighting matrices is often required in order to achieve desired transient response. Thus a design method which enables us to construct an optimal regulator with specified closed-loop poles is desired.

Several design methods have been reported in which the desired pole placement is achieved by shifting either a single real pole or a pair of complex conjugate poles at a time. Early results⁽⁴⁾⁽⁵⁾ have a serious limitation on the achievable closed-loop poles, due to a restricted form of the weighting matrix used in the design procedure. This limitation is relaxed to some extent by a method in which the weighting matrix is chosen so that the closed-loop poles may be shifted along the real axis⁽⁶⁾. Then another method was presented, where both real part and imaginary part of the closed-loop poles can be arbitrarily specified⁽⁷⁾. However, this result is based on the stable regulator problem rather than the optimal regulator problem, where the stability of the closed-loop system is given a higher priority over the minimization of the performance criterion. This may result in the loss of optimality guaranteed in the usual linear quadratic regulator problem.

The present paper develops a method for constructing a linear quadratic regulator that achieves the desired pole placement while satisfying the optimality. It follows the basic concept of shifting a single real pole or a pair of complex conjugate poles at a time. A weighting matrix is constructed in such a way that the desired pole

location is achieved by the optimal feedback gain corresponding to the weighting matrix of the performance criterion.

The feasibility of the pole shifting can be verified in each step of the design procedure, since this method utilizes the complete region of assignable closed-loop poles⁽⁸⁾.

For multi-input systems, feedback gain corresponding to a specific set of closed-loop poles is not unique. Thus the optimality of the closed-loop system may be regarded as an additional specification for the selection of a particular feedback gain among those that result in the identical pole placement.

2. Successive pole assignment

Consider a continuous-time linear time-invariant system described as

$$\dot{x}(t) = Ax(t) + Bu(t) \dots\dots\dots (1)$$

$$y(t) = Cx(t) \dots\dots\dots (2)$$

where $x \in R^n$, $u \in R^m$, $y \in R^r$, and A, B, C are matrices of appropriate dimension. Here, we assume for simplicity that the pair (A, B) is controllable and the coefficient matrix A has distinct eigenvalues. We will use state feedback to control the system.

Now, select a specific mode of this system, and extract the specified mode by an appropriate linear transformation. The above assumption guarantees that the selected mode is controllable, and we can choose a non-singular matrix M so that

$$M^{-1}AM = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \dots\dots\dots (3)$$

$$M^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \dots\dots\dots (4)$$

and either 1-by-1 or 2-by-2 matrix A_{11} represents the specified mode.

The quadratic performance criterion for this system

$$J = \int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t)\} dt \dots\dots (5)$$

shall be transformed in the same manner. Here the weighting matrix Q is to be constructed according to the pole assignment. Let Q_{11} be a positive semidefinite matrix with the same size as A_{11} , and set the weighting matrix Q that satisfies

$$Q = (M^{-1})^T \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} M^{-1} \dots\dots\dots (6)$$

Then the eigenvalue of A_{11} can be shifted while keeping all other eigenvalues of A unchanged⁽⁴⁾.

Thus, appropriate selection of the weighting matrix Q through Q_{11} is crucial in the design of optimal regulators with prescribed closed-loop poles. The selection of weighting matrix R is arbitrary from this point of view, and R could be used as a scaling factor for the input channels. Scaling R for single-input systems has no effect, since it will only result in the same amount of scaling on Q .

Hereafter, we will restrict the selection of Q_{11} to those matrices such that the observability of the pair $(Q_{11}^{-1/2}, A_{11})$ is satisfied. Thus a particular selection of $Q_{11} = 0$ is always excluded even if we refer to Q_{11} as a positive semidefinite matrix.

Regarding the weighting matrix R , let us define a matrix V by

$$V := BR^{-1}B^T \dots\dots\dots (7)$$

The above transformation matrix M applied to V yields $M^{-1}V(M^{-1})^T$, which is partitioned as

$$M^{-1}V(M^{-1})^T = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix} \dots\dots\dots (8)$$

where V_{11} has the same size as A_{11} and Q_{11} .

Now, we can concentrate on either first or second order system represented by the matrices A_{11} , Q_{11} , and V_{11} , or, equivalently, A_{11} , B_1 , Q_{11} , and R . Once the desired pole positioning is accomplished by appropriate weight selection, another mode can be selected and shifted in the same manner. The design procedure⁽⁴⁾ is summarized in the following.

Step 1 Choose a transformation matrix M so that the partitioned matrices A_{11} and V_{11} represent either a real pole or a pair of complex conjugate poles to be shifted.

Step 2 Find a weighting matrix Q_{11} with which the desired pole positioning is accomplished.

Step 3 Calculate the weighting matrix Q and the corresponding optimal feedback gain F for the whole system, then form a closed-loop system with F .

Step 4 Go back to Step 1 while there are remaining poles to be shifted.

Step 5 Calculate the sum of the matrices Q and F in each step to obtain the weighting matrix and optimal feedback gain which achieve the desired pole positioning.

In the following sections we are mainly concerned with Step 2 of the procedure, where the region of admissible poles is determined. Selection of the transformation matrix M in Step 1 is also treated in more detail, which is

relevant to the derivation of the result.

It may be noted that there is no restriction in the location of closed-loop poles, if the stable regulator problem were to be employed in the design. But then the significance of optimality may be obscured, especially for single-input systems, where the feedback gain is uniquely determined by the location of the closed-loop poles. Thus we adhere to optimality of the closed-loop poles in the following.

3. Relationship between the weighting matrix and closed-loop poles

When a real pole is to be shifted, the matrices A_{11} , Q_{11} , V_{11} reduce to scalars. In this case the assignable region of optimal closed-loop poles is readily determined. It turns out that a real pole, either stable or unstable, can only be shifted along the real axis within the left-half plane, and that the absolute value of the closed-loop pole is larger than that of the open-loop pole.

We assume in the following that a pair of complex conjugate poles are to be shifted, where A_{11} , Q_{11} , V_{11} are 2-by-2 matrices. The system to be considered here is described by

$$\dot{x}_1(t) = A_{11}x_1(t) + B_1u(t) \dots\dots\dots (9)$$

where $x_1 \in \mathbb{R}^2$, $u \in \mathbb{R}^m$, and A_{11} has a pair of complex conjugate poles. It may be noted that this system may have more inputs than states, since m may be greater than two, depending on the original system. The corresponding performance criterion to be minimized is

$$J = \int_0^\infty \{x_1^T(t)Q_{11}x_1(t) + u^T(t)Ru(t)\} dt \dots (10)$$

Now, we have

$$V_{11} = B_1R^{-1}B_1^T \dots\dots\dots (11)$$

according to (4), (7), and (8). The problem is to determine the region of optimal closed-loop poles attained by proper choice of Q_{11} for a given set of A_{11} and V_{11} .

Let the open-loop poles of the specified mode be $\alpha \pm j\beta$ with $\beta \neq 0$. We claim the following proposition.

Proposition 1 There exists a nonsingular matrix M such that A_{11} in (3) and V_{11} in (8) take the form

$$A_{11} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, V_{11} = v_0 \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} \dots\dots (12)$$

where $v_0 > 0$ and $0 \leq v \leq 1$.

Proof If we use the real part and imaginary part of the eigenvectors of A corresponding to the eigenvalues $\alpha \pm j\beta$ as the first two columns of the transformation matrix, A_{11} results. Let this transformation matrix be M_1 . Other columns of the transformation matrix are irrelevant.

The matrix V_{11} is symmetric and positive semidefinite in general. Thus it can be diagonalized and arbitrary ordering of the eigenvalues is possible with an orthogonal transformation matrix. Application of a second-order orthogonal matrix to A_{11} yields either A_{11} itself or A_{11}^T , but A_{11}^T is within the form of A_{11} , since the sign of β

is not specified. Append identity matrix to this orthogonal matrix to form a block diagonal matrix with the size of A , and let us denote this matrix as M_2 .

Then the transformation matrix $M := M_1 M_2$ yields the claimed result. \square

Scaling the weighting matrices Q and R of the performance criteria by the same factor v_0 does not affect the optimal feedback gain and the closed-loop poles. Thus, we can assume that $v_0 = 1$, as far as the assignable region of the closed-loop poles is concerned. The scaling factor v_0 is to be adjusted after calculating the weighting matrix Q_{11} for the specified closed-loop poles. Thus, we can concentrate on the specific forms of 2-by-2 matrices A_{11} and V_{11} in the sequel without loss of generality.

Consider a Hamilton matrix

$$H = \begin{bmatrix} A_{11} & V_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} \dots \dots \dots (13)$$

associated with the regulator problem of the second-order system, and let the entries of Q_{11} be described as

$$Q_{11} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \dots \dots \dots (14)$$

Then the characteristic equation of H has the form

$$s^4 + c_2 s^2 + c_0 = 0 \dots \dots \dots (15)$$

The coefficients c_2 and c_0 turn out to be

$$c_2 = 2(\beta^2 - \alpha^2) - \bar{c}_2 \dots \dots \dots (16)$$

$$c_0 = (\alpha^2 + \beta^2)^2 + \bar{c}_0 \dots \dots \dots (17)$$

where

$$\bar{c}_2 := q_1 + vq_3 \dots \dots \dots (18)$$

$$\bar{c}_0 := (\alpha^2 + v\beta^2)q_1 + 2(1-v)\alpha\beta q_2 + (v\alpha^2 + \beta^2)q_3 + v(q_1 q_3 - q_2^2) \dots \dots \dots (19)$$

Here, \bar{c}_2 and \bar{c}_0 represent the terms which depend on the weighting matrix Q_{11} .

Remark 1 Remember that the sign of α and β are not restricted, where $\alpha > 0$ implies that the open-loop pole is unstable. We could restrict β to be either positive or negative, if we did not require the ordering of the diagonal element of V_{11} . But then $(1-v)$ could become negative. Furthermore, the sign of q_2 is undetermined with the requirement of Q_{11} being positive semidefinite. Thus the sign of $(1-v)\alpha\beta q_2$ is indefinite. This implies that the sign of α and/or β does not affect the range of \bar{c}_0 , hence the admissible region of optimal closed-loop poles. In fact, it only affects the value of the weighting matrix Q for the whole system. Similarly, the scaling factor v_0 in V_{11} only affects Q , and not the region of admissible closed-loop poles.

The eigenvalues of the Hamilton matrix H are symmetric with respect to the imaginary axis, or equivalently in this case, with respect to the origin, and those eigenvalues in the left-half plane correspond to the optimal closed-loop poles.

Let the weighting matrix Q_{11} correspond to the optimal closed-loop poles $\alpha_c \pm j\beta_c$. Then the coefficients c_2 and c_0 are described as

$$c_2 = 2(\beta_c^2 - \alpha_c^2) \dots \dots \dots (20)$$

$$c_0 = (\alpha_c^2 + \beta_c^2)^2 \dots \dots \dots (21)$$

This leads to the relationship between the weighting matrix and the optimal closed-loop poles. The region of assignable optimal closed-loop poles is characterized in terms of the region of the coefficients c_2 and c_0 subject to positive semidefiniteness of Q_{11} .

When α_c and β_c are specified as a desired closed-loop location, whether it can be optimal or not depends on the existence of a positive semidefinite matrix Q_{11} such that the coefficients c_2 , c_0 of (16), (17) and (20), (21) coincide.

4. The region of closed-loop poles

We continue the development on the second-order system described by A_{11} , V_{11} , and Q_{11} .

First, let us assume that the 2-by-2 weighting matrix Q_{11} is restricted to be singular. In this case Q_{11} can be described in dyadic form as

$$Q_{11} = \begin{bmatrix} \rho \cos^2 \theta & \rho \cos \theta \sin \theta \\ \rho \cos \theta \sin \theta & \rho \sin^2 \theta \end{bmatrix} \dots \dots \dots (22)$$

where $\rho > 0$ and $0 \leq \theta < \pi$. Recall that it suffices to consider Q_{11} of the form $Q_{11} = C^T C$ with row dimension of C equal to the dimension of u ⁽³⁾. Thus restricting the weighting matrix Q_{11} to be singular does not result in loss of generality for single-input systems. This also implies that a separate argument is needed for multi-input systems, which will be treated later.

Now the parameters \bar{c}_2 and \bar{c}_0 can be expressed in terms of ρ and θ as

$$\bar{c}_2 = \frac{\rho}{2} \{ (1+v) + (1-v) \cos 2\theta \} \dots \dots \dots (23)$$

$$\bar{c}_0 = \frac{\rho}{2} (\alpha^2 + \beta^2) \{ (1+v) + (1-v) \cos(2\theta + \phi) \} \dots \dots \dots (24)$$

where ϕ satisfies

$$\sin \phi = \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \quad \cos \phi = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \dots \dots \dots (25)$$

It can be shown that \bar{c}_2 and \bar{c}_0 are positive for arbitrary ρ and θ satisfying the observability condition.

This leads to the following result.

Theorem 1 Let σ be given by

$$\sin \sigma = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \quad \cos \sigma = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \dots \dots (26)$$

Then the range of \bar{c}_2 and \bar{c}_0 for singular weighting matrices Q_{11} is given by

$$k_1 \leq \frac{\bar{c}_0}{\bar{c}_2} \leq k_2 \dots \dots \dots (27)$$

where the boundary values k_1 and k_2 are attained when θ satisfies

$$\sin(2\theta + \sigma) = -\frac{1-v}{1+v} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \dots\dots\dots (28)$$

Proof Note that the ratio $\bar{\tau}_0/\bar{\tau}_2$ does not depend on the value of ρ . Differentiating the ratio by θ leads to the claimed result. \square

Next, we consider the region of $\bar{\tau}_2$ and $\bar{\tau}_0$ without restricting Q_{11} to be singular. This is necessary for deriving the results for multi-input systems, as mentioned before. To this end we will fix the value of $\bar{\tau}_2$ and investigate the admissible region for the other parameter $\bar{\tau}_0$.

Lemma 1 When the value of $\bar{\tau}_2$ is fixed at $\bar{\tau}_2 = \bar{\tau}$, the other parameter $\bar{\tau}_0$ attains its minimum value when Q_{11} is singular.

Proof It can be verified from the observation that the value of $v(q_1q_3 - q_2^2)$ in (19) is minimized when $\det Q_{11} = 0$. \square

The next condition applies to multi-input systems only.

Lemma 2 Suppose $v \neq 0$ and $\bar{\tau}_2$ is fixed at $\bar{\tau}$. Let the entries of the weighting matrix Q_{11} be

$$q_1 = \frac{v\bar{\tau} - (1-v^2)\beta^2}{2v} \dots\dots\dots (29)$$

$$q_2 = \frac{(1-v)\alpha\beta}{v} \dots\dots\dots (30)$$

$$q_3 = \frac{v\bar{\tau} + (1-v^2)\beta^2}{2v^2} \dots\dots\dots (31)$$

Then $\bar{\tau}_0$ cannot be maximized by a singular weighting matrix iff the above Q_{11} is positive definite.

Proof It can be verified that maximization of $\bar{\tau}_0$ with respect to the entries of Q_{11} subject to $\bar{\tau}_2 = \bar{\tau}$ always result in (29)–(31), provided the positive semidefiniteness of Q_{11} is neglected. Then taking the positive semidefinite constraint into account leads to the claimed result. \square

The positive definiteness of the matrix Q_{11} in the above lemma generally depends on the value of $\bar{\tau}$ for the given system. It does become positive definite if $\bar{\tau}$ is sufficiently large.

Lemma 3 Suppose $v = 0$. Then $\bar{\tau}_0$ can be made arbitrary large for a fixed value of $\bar{\tau}_2$ with a positive semidefinite matrix Q_{11} .

Proof Substituting $v = 0$ into (18) and (19) yields

$$\bar{\tau}_2 = q_1 \dots\dots\dots (32)$$

$$\bar{\tau}_0 = \alpha^2q_1 + 2\alpha\beta q_2 + \beta^2q_3 \dots\dots\dots (33)$$

from which the result is obvious. \square

The results obtained so far are summarized in the next theorem, in which the region of assignable optimal poles is given in terms of the parameter $\bar{\tau}_0$ for fixed values of $\bar{\tau}_2$.

Theorem 2 Fix the value of $\bar{\tau}_2$ at $\bar{\tau}_2 = \bar{\tau}$. The minimum value of $\bar{\tau}_0$ is always given by $k_1\bar{\tau}$, where k_1 is the value introduced in Theorem 1. If $v = 0$, $\bar{\tau}_0$ can be arbitrarily large. If $v \neq 0$, the maximum value of $\bar{\tau}_0$ is determined either by Lemma 2 or given by $k_2\bar{\tau}$, depending on the positive definiteness of the matrix Q_{11} whose entries are given by (29)–(31).

Proof The result follows naturally from Theorem 1 and Lemmas 1–3. \square

When a set of open-loop poles and that of desired closed-loop poles are given, we can readily calculate the corresponding values of $\bar{\tau}_2$ and $\bar{\tau}_0$. Thus we can verify if it is admissible, i.e., whether there exists a positive semidefinite weighting matrix Q_{11} which achieves the desired pole shifting.

The boundary of admissible values of $\bar{\tau}_2$ and $\bar{\tau}_0$ can be mapped to the boundary of optimal poles by solving the characteristic equation of H , which gives a graphical representation of the admissible closed-loop poles. This can be utilized in Step 2 of the design procedure, if the initially specified closed-loop poles does not satisfy the optimality condition, and the selection of a feasible location is to be determined by trial and error.

5. Numerical Example

Let us consider a sixth-order system described by the following coefficient matrices⁽⁶⁾.

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \dots\dots (34)$$

This is an unstable system with the following set of open-loop poles.

$$\{-0.7869 \pm j1.0096, -1.5462, -2.6511, -3.9776, 0.7486\}$$

We first stabilize the system with the weighting matrices $Q_0 = \text{diag}\{1, 1, 1, 0, 0, 0\}$ and $R = I$. The optimal feedback gain F_0 corresponding to these weighting matrices turns out to be

$$F_0 = \begin{bmatrix} 0.1891 & 0.4159 & 0.9499 \\ 0.2138 & 0.4171 & 0.8275 \\ & 0.3692 & 0.0695 & 0.8275 \\ & 0.4044 & 0.0407 & 1.0255 \end{bmatrix} \dots\dots (35)$$

and the eigenvalues of $A - BF_0$ are as follows:

$$\{-0.7699 \pm j1.0716, -1.0297, -1.7642, -2.6565, -3.9851\}$$

It should be noted that the initial feedback F_0 is not essential to our design method. It is solely for the sake of comparison with the result shown in the reference⁽⁶⁾.

Now we consider shifting a mode at $-0.7699 \pm j1.0716$ by -0.3 , again in accordance with the reference⁽⁶⁾. This mode can be extracted by using the transformation matrix

$$M = \begin{bmatrix} -0.1568 & -0.3132 & -0.2428 \\ -0.1341 & -0.2113 & 0.8518 \\ -0.1925 & 0.0403 & -0.2265 \\ -0.4663 & -0.2333 & -0.1615 \\ 0.2768 & -0.3420 & -0.3698 \\ 0.5316 & 0.1671 & -0.0364 \\ 0.7380 & 0.0226 & 0.0200 \\ 0.0793 & 0.3460 & -0.2042 \\ -0.1469 & 0.7456 & -0.6459 \\ 0.2925 & -0.0070 & -0.6535 \\ -0.5639 & -0.3406 & 0.2236 \\ -0.1547 & -0.4559 & -0.2523 \end{bmatrix} \dots (36)$$

and we obtain

$$A_{11} = \begin{bmatrix} -0.7699 & 1.0716 \\ -1.0716 & -0.7699 \end{bmatrix} \dots (37)$$

$$B_1 = \begin{bmatrix} -0.7041 & 1.3440 \\ 0.1368 & -0.0717 \end{bmatrix} \dots (38)$$

$$V_{11} = 2.3021 \begin{bmatrix} 1 & 0 \\ 0 & 0.0104 \end{bmatrix} \dots (39)$$

namely, $\alpha = -0.7699$, $\beta = 1.0716$, $v_0 = 2.3021$, and $v = 0.0104$ in (12).

Substituting these values together with the parameters of the specified closed-loop system $\alpha_c = -1.0699$ and $\beta_c = 1.0716$ into (16), (17), (20), and (21) results in

$$\bar{c}_2 = 1.1039, \quad \bar{c}_0 = 2.2265 \dots (40)$$

The feasibility of this particular pole shifting can be verified by applying Theorems 1 and 2. It turns out that $\sigma = -0.6230$ in (26) and either $\theta = 0.6413$ or $\theta = 1.5795$ satisfies (28) in Theorem 1. The values of θ together with the value of $\phi = 1.8955$ in (25) yield the boundary values $k_1 = 0.0322$ and $k_2 = 94.06$ in (27). The ratio $\bar{c}_0/\bar{c}_2 = 2.0169$ is clearly within the feasible range.

Now, let us calculate the corresponding weighting matrix Q_{11} in (22). Solving the equations (23) and (24) for ρ and θ yields the set of solutions $\rho = 1.5064$, $\theta = -0.5464$ and $\rho = 5.4982$, $\theta = 1.1167$, corresponding to singular weighting matrices

$$Q_{11} = \begin{bmatrix} 1.0997 & -0.6688 \\ -0.6688 & 0.4067 \end{bmatrix} \dots (41)$$

and

$$Q_{11} = \begin{bmatrix} 1.0579 & 2.1673 \\ 2.1673 & 4.4403 \end{bmatrix} \dots (42)$$

respectively. Here, recall that the relationship between the closed-loop poles and the weighting matrix Q_{11} was analyzed with the assumption of $v_0 = 1$, and that this is not the case in our example. When the scaling factor v_0 takes a general value, the actual weighting matrix corresponding to the coefficient matrices A_{11} and $V_{11} = B_1 R^{-1} B_1^T$ is Q_{11}/v_0 . With this adjustment, either of the above weighting matrices Q_{11} leads to the

desired closed-loop poles of the second-order system described by A_{11} and B_1 .

Furthermore, the weighting matrix Q_{11} need not be singular, as far as it is positive semidefinite and satisfies the equations (23) and (24). It can be verified that

$$Q_{11} = \begin{bmatrix} 1.0785 & 0.7491 \\ 0.7491 & 2.4069 \end{bmatrix} \dots (43)$$

is such a nonsingular weighting matrix that yields the specified closed-loop poles. Again, the scaling factor v_0 needs to be taken into account.

The weighting matrix Q for the full-order system can be calculated by (6). The result for the last example of Q_{11}/v_0 is

$$Q_1 = \begin{bmatrix} 0.8077 & 0.6732 & 0.2289 \\ 0.6732 & 0.7229 & 0.0490 \\ 0.2289 & 0.0490 & 0.1892 \\ -0.1799 & 0.2002 & -0.3579 \\ 0.9362 & 1.1234 & -0.0353 \\ 0.2286 & -0.1803 & 0.3898 \\ -0.1799 & 0.9362 & 0.2286 \\ 0.2002 & 1.1234 & -0.1803 \\ -0.3579 & -0.0353 & 0.3898 \\ 0.7980 & 0.5341 & -0.8536 \\ 0.5341 & 1.8128 & -0.5214 \\ -0.8536 & -0.5214 & 0.9147 \end{bmatrix} \dots (44)$$

and the optimal feedback gain corresponding to this weighting matrix is

$$F_1 = \begin{bmatrix} 0.3013 & 0.1841 & 0.1441 \\ 0.3682 & 0.1534 & 0.2388 \\ -0.2121 & 0.2071 & 0.2388 \\ -0.4141 & 0.1014 & 0.4559 \end{bmatrix} \dots (45)$$

Finally, the optimal regulator problem for the original, unstable open-loop system described by the coefficient matrices A and B is considered. Setting the weighting matrices of the performance index to be $Q := Q_0 + Q_1$ and $R = I$, the optimal feedback gain is given by $F := F_0 + F_1$, and the eigenvalues of $A - BF$ are

$$\{-1.0699 \pm j1.0716, -1.0297, -1.7642, -2.6565, -3.9851\}$$

as specified.

6. Conclusions

A design method for the linear quadratic optimal regulator with specified closed-loop poles is presented, in which a real pole or a pair of complex conjugated poles are shifted at a time. The complete region of admissible closed-loop is exploited in the design procedure. This is accomplished by a detailed examination of the relationship between the weighting matrix and the optimal poles, especially for complex conjugate poles. Thus the feasibility of a particular pole shifting can be verified, using the given parameters. Although only a single mode is to be shifted in each step of the design procedure, a single weighting matrix for the whole system

which attains the desired pole placement in one step is obtained as a result. Discrete-time systems can be treated in a similar manner. However, a symplectic matrix must be used instead of the Hamilton matrix, and the corresponding formulas become more involved.

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