Paper

State Space Model Identification Using Subspace Extraction via Schur Complement

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In this paper, another interpretation for Subspace-based State Space System IDentification (4SID) methods by subspace extraction via Schur complement is presented. In the ordinary MIMO Output-Error State space model identification (MOESP) algorithm, it is shown that the estimate of the extended observability matrix is obtained from Schur complement matrix, which is derived from the matrix consisted of the Hankel matrices of input-output data. The proposed method is applied similarly to Instrumental Variable (IV) based 4SID methods. A feature of our method is that the same procedure can be applicable to both cases, MOESP and IV-based one. We also propose a recursive computation based on the subspace extraction via Schur complement. Finally, a numerical example illustrates the proposed algorithm.

Keywords: Schur complement, 4SID method, instrumental variables method, recursive computation, system identification

1. Introduction

The 4SID methods have attracted much attention because of being essentially suitable for the identification of MIMO systems. The methods directly realize system matrix of state space model from input-output data without intermediate expression such as impulse response or difference equation. The methods are characterized by the determination of the extended observability matrix from input-output data. The MOESP algorithm [1], [2] is known as the ordinary 4SID method. The QR factorization and the singular value decomposition (SVD) are the principal computational tools. We consider the QR factorization in the MOESP algorithm, and show alternative derivation of the estimate of the extended observability matrix by subspace extraction via Schur complement [3]. Then we also propose the above derivation for IV-based 4SID method. A relationship between the least squares residual and the Schur complement matrix obtained from input-output data is shown, and we propose a recursive formula for the error covariance matrix in the 4SID method.

The paper is organized as follows. In the next section, we present the problem statement and the ordinary MOESP algorithm. Section 3 provides another interpretation of the 4SID method, for deterministic and noisy cases. In section 4, we describe a recursive computation by using the results obtained in section 3. A brief discussion on the numerical example of the recursive algorithm is presented in section 5. Finally section 6 concludes this paper.

2. Preliminaries

2.1 System description We consider a discrete time linear time-invariant system by the following state space equation:

$$x_{k+1} = Ax_k + Bu_k \quad \cdots\cdots (1)$$

$$y_k = Cx_k + Du_k \quad \cdots\cdots (2)$$

where $x_k \in \mathrm{R}^n$ is the state vector, $u_k \in \mathrm{R}^m$ is the input and $y_k \in \mathrm{R}^l$ is the output, respectively. The unknown system matrices A, B, C, and D have appropriate dimensions. It is assumed that the model is minimal, that is, the system is completely reachable and observable.

The Hankel matrix $U_{k,i,N}$ of $\{u_k\}$ is defined as follows;

$$U_{k,i,N} := \begin{bmatrix} u_k & u_{k+1} & \cdots & u_{k+N-1} \\ u_{k+1} & u_{k+2} & \cdots & u_{k+N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k+i-1} & u_{k+i} & \cdots & u_{k+N+i-2} \end{bmatrix} \quad (3)$$

and $Y_{k,i,N}$ is defined in a similar way. The index triplet k, i, N of the Hankel matrix determines the dimensions of this matrix, as well as which part of the data batch is stored in the Hankel matrix. In relation to the order of the system n the pair i, N satisfy $i > n$ and $N \gg n$. We define the state vector sequence as

$$X_{k,N} := [x_k \ x_{k+1} \ \cdots \ x_{k+N-1}] \quad \cdots\cdots (4)$$

Then, we obtain the following relationship between the data matrices $X_{k,N}$, $U_{k,i,N}$ and $Y_{k,i,N}$:

$$Y_{k,i,N} := \Gamma_i X_{k,N} + H_i U_{k,i,N} \quad \cdots\cdots (5)$$

where

$$\Gamma_i := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, \quad \cdots\cdots (6)$$

$$H_i := \begin{bmatrix} D & & & O \\ CB & D & & \\ \vdots & \ddots & \ddots & \\ CA^{i-2}B & \cdots & CB & D \end{bmatrix}. \quad \cdots\cdots (7)$$

Then, Γ_i is called the extended observability matrix. We will omit the subscripts for U and Y unless otherwise mentioned.

An estimated realization of the system matrices are denoted by

$$[A_T,\ B_T,\ C_T,\ D_T] = [TAT^{-1},\ TB,\ CT^{-1},\ D]$$

where T is a nonsingular matrix. Let the input u_k be such that the following condition is satisfied.

$$\text{rank}\begin{bmatrix} U \\ X \end{bmatrix} = mi + n \quad \cdots\cdots (8)$$

2.2 The ordinary 4SID algorithm In this section, we describe the ordinary 4SID algorithm. An efficient implementation of the basic 4SID algorithm, so-called MOESP algorithm [1], is proposed by Verhaegen *et al.* The ordinary MOESP algorithm is described as follows.

MOESP algorithm:

step1 Construct the Hankel matrices U and Y defined in (3), and achieve a data compression via the following QR factorization:

$$\begin{bmatrix} U \\ Y \end{bmatrix} = \begin{bmatrix} R_{11} & O \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \quad \cdots\cdots (9)$$

where R_{11} and R_{22} are lower triangular, and $Q_1Q_1^T = I_{mi}$, $Q_2Q_2^T = I_{li}$, $Q_1Q_2^T = O$.

step2 Compute SVD of the matrix R_{22} given in (9), i.e.,

$$R_{22} = [E_n\ E_n^{\perp}] \begin{bmatrix} \Sigma_n & O \\ O & \Sigma_2 \end{bmatrix} \begin{bmatrix} F_n^T \\ (F_n^{\perp})^T \end{bmatrix} \quad \cdots (10)$$

The dimension of Σ_n is equal to the one of the system.

step3 Using the matrix E_n given in (10), solve the set of equations for A_T and C_T

$$C_T = E_n(1{:}l,\ :) \quad \cdots\cdots (11)$$

$$E_n^{(1)} A_T = E_n^{(2)} \quad \cdots\cdots (12)$$

where $E_n(1{:}l,\ :)$ denotes the first l rows of E_n, $E_n^{(1)}$ is the submatrix composed of the first $(i-1)l$ rows of the matrix E_n and $E_n^{(2)}$ is constructed by the last rows in a similar way. Equation (12) express the *shift-invariance* property in the estimate E_n of Γ_i.

step4 Solve the following equation for B_T and D.

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_i \end{bmatrix} = \Psi \begin{bmatrix} D \\ B_T \end{bmatrix} \quad \cdots\cdots (13)$$

where ξ_j, ψ_j, and Ψ are defined by the following relations:

$$[\ \xi_1\ \ \xi_2\ \ \cdots\ \ \xi_i\ ] := (E_n^{\perp})^T R_{21} R_{11}^{-1} \quad \cdots\cdots (14)$$

$$[\ \psi_1\ \ \psi_2\ \ \cdots\ \ \psi_i\ ] := (E_n^{\perp})^T \quad \cdots\cdots (15)$$

$$\Psi := \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_i \\ \psi_2 & & \psi_i & \\ \vdots & \psi_i & & \\ \psi_i & & & O \end{bmatrix} \begin{bmatrix} I_l & O \\ O & E_n^{(1)} \end{bmatrix} \cdot (16)$$

The size of ξ_j $(1 \le j \le i)$ and ψ_j $(1 \le j \le i)$ is $(li-n) \times m$ and $(li-n) \times l$, respectively.

3. Subspace extraction via Schur complement

We show a new interpretation of the 4SID method by subspace extraction via Schur complement [3]. In the MOESP algorithm, the QR factorization and the SVD are used for estimating of the extended observability matrix. We present alternative derivation of the estimate by using Schur complement of input-output data matrix. It is shown that the proposed procedure can treat IV-based 4SID algorithm as well as an elementary 4SID algorithm.

The Schur complement is defined as follows.

Definition:

Suppose we partition A represented by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \ (17)$$

Assume that A_{11} is nonsingular. Then the matrix $S = A_{22} - A_{21}A_{11}^{-1}A_{12}^T$ is called the Schur complement of A_{11} in A.

3.1 Noise-free case We consider the following matrix constructed from the Hankel matrices of input-output data.

$$\begin{bmatrix} U \\ Y \end{bmatrix} [\ U^T\ \ Y^T\ ] = \begin{bmatrix} UU^T & UY^T \\ YU^T & YY^T \end{bmatrix} \quad \cdots (18)$$

The Schur complement denoted by S_1 of UU^T in (18) is represented by

$$\begin{aligned} S_1 &= YY^T - YU^T(UU^T)^{-1}UY^T \\ &= Y\Pi_U^{\perp}Y^T \quad \cdots\cdots (19) \end{aligned}$$

where $\Pi_U = U^T(UU^T)^{-1}U$ and $\Pi_U^{\perp} = I - \Pi_U$. Using the matrix given in (9), then S_1 in (19) can be rewritten by

$$\begin{aligned} S_1 &= R_{22}Q_2Q_2^T R_{22}^T \\ &= R_{22}R_{22}^T. \quad \cdots\cdots (20) \end{aligned}$$

From results of (19) and (20), we have

$$Y\Pi_U^{\perp}Y^T = R_{22}R_{22}^T \quad \cdots\cdots (21)$$

It is clear that the eigenvalues of $R_{22}R_{22}^T$ coincide with the left singular vectors of $Y\Pi_U^{\perp}Y^T$, and we see that the matrix E_n in equation (10) is obtained by computing SVD of $Y\Pi_U^{\perp}Y^T$. For another variation, the Observability Range Space Extraction (ORSE) method [4] is presented by Liu. The ORSE method requires the pretreatment of input-output data in order to obtain the estimate of the extended observability matrix. It is remarkable that the proposed method does not require these treatment.

3.2 **Noisy case** It is assumed that the output of the system is perturbed by the noise v_k, where the input u_k and the noise v_k are independent. Then the output equation reads

$$z_k = y_k + v_k \quad \cdots\cdots (22)$$

Hankel matrices of $\{z_k\}$ and $\{v_k\}$ is represented by Z and V, then we have

$$Z = Y + V \quad \cdots\cdots (23)$$

Using a matrix $\widetilde{R}_{22}$ yielded by computing the QR factorization of a matrix $\left[U^T \ Z^T\right]^T$ in a similar way as in (9), the following relation is obtained from (21)

$$\lim_{N\to\infty} \frac{1}{N}\widetilde{R}_{22}\widetilde{R}_{22}^T = \frac{1}{N}Y\Pi_U^{\perp}Y^T + R_{vv} \quad \cdots\cdots (24)$$

where R_{vv} is a covariance matrix of v_k. From equation (24) the estimate of the extended observability matrix is not obtained by computing singular value decomposition of $Z\Pi_U^{\perp}Z^T$ asymptotically. Therefore we introduce an instrumental variable Φ, satisfying the following conditions:

$$\lim_{N\to\infty} \frac{1}{N}\Phi V^T = O, \ (25)$$

$$\text{rank}\begin{bmatrix} U \\ \Phi \end{bmatrix} = mi + p \quad \cdots\cdots (26)$$

where the size of Φ is $p \times N$ and $p > n$.

A MOESP algorithm with the instrumental variable Φ (IV-MOESP) [2] is described as follows.

IV-MOESP algorithm:

step1$'$ Compute QR factorization of a matrix consisted of input-output data U, Z and an instrumental variable Φ, i.e.

$$\begin{bmatrix} U \\ Z \\ \Phi \end{bmatrix} = \begin{bmatrix} \widetilde{R}_{11} & & O \\ \widetilde{R}_{21} & \widetilde{R}_{22} & \\ \widetilde{R}_{31} & \widetilde{R}_{32} & \widetilde{R}_{33} \end{bmatrix} \begin{bmatrix} \widetilde{Q}_1 \\ \widetilde{Q}_2 \\ \widetilde{Q}_3 \end{bmatrix} \quad \cdots (27)$$

step2$'$ Apply SVD to $\widetilde{R}_{22}\widetilde{R}_{32}^T$ given in (27) as follows

$$\widetilde{R}_{22}\widetilde{R}_{32}^T = [E_n \ E_n^{\perp}] \begin{bmatrix} \Sigma_n & O \\ O & \Sigma_2 \end{bmatrix} \begin{bmatrix} F_n^T \\ (F_n^{\perp})^T \end{bmatrix} \quad (28)$$

step3$'$ Using E_n, $E_n^{\perp}$, $\widetilde{R}_{11}$, and $\widetilde{R}_{21}$ yielded in step1$'$ and step2$'$, compute the quadruple of the system matrices $[A, \ B, \ C, \ D]$ in a similar way in **MOESP algorithm**. We consider a matrix $\widehat{Z}$ which is a linear combination of U and Φ. This is represented by

$$\begin{aligned} \widehat{Z} &:= L_1U + L_2\Phi \\ &= \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} U \\ \Phi \end{bmatrix} \\ &= L\Omega \quad \cdots\cdots (29) \end{aligned}$$

where $L := [L_1 \ L_2]$ and $\Omega := \left[U^T \ \Phi^T\right]^T$. Since the condition (26) is satisfied, $\widehat{L}$ which minimize $\| Z - \widehat{Z} \|_F^2$ for Ω exists uniquely. The notation $\| \cdot \|_F$ denotes the Frobenius norm. Then $\widehat{L}$ is represented by

$$\widehat{L} = Z\Omega^T(\Omega\Omega^T)^{-1}. \quad \cdots\cdots (30)$$

From the result of (30), we substitute $\widehat{L}$ for L in (29) to obtain

$$\begin{aligned} \widehat{Z} &= Z\Omega^T(\Omega\Omega^T)^{-1}\Omega \\ &= Z\Pi_{\Omega} \quad \cdots\cdots (31) \end{aligned}$$

where $\Pi_{\Omega} = \Omega^T(\Omega\Omega^T)^{-1}\Omega$. We consider a matrix $\left[U^T \ \widehat{Z}^T\right]^T$, then $\widehat{Z}$ and V are uncorrelated on the assumption. Therefore we have

$$\lim_{N\to\infty} \frac{1}{N}V\left[\ U^T \quad \widehat{Z}^T \ \right] = O \quad \cdots\cdots (32)$$

Suppose the following matrix as in (18)

$$\begin{aligned} \begin{bmatrix} U \\ \widehat{Z} \end{bmatrix}\left[U^T \ \widehat{Z}^T\right] &= \begin{bmatrix} U \\ Z \end{bmatrix}\Pi_{\Omega}\left[U^T \ Z^T\right] \\ &= \begin{bmatrix} UU^T & UZ^T \\ ZU^T & Z\Pi_{\Omega}Z^T \end{bmatrix} \quad \cdots (33) \end{aligned}$$

where $U\Pi_{\Omega} = U$. Thus the Schur complement denoted by S_2 of UU^T in (33) is represented by

$$\begin{aligned} S_2 &= Z(\Pi_{\Omega} - U^T(UU^T)^{-1}U)Z^T \\ &= \widehat{Z}\Pi_U^{\perp}\widehat{Z}^T. \quad \cdots\cdots (34) \end{aligned}$$

From the definition of the Ω, Π_{Ω} can be rewritten as

$$\Pi_{\Omega} = \Pi_U + \Pi_U^{\perp}\Phi^T(\Phi\Pi_U^{\perp}\Phi^T)^{-1}\Phi\Pi_U^{\perp}. \quad \cdots\cdots (35)$$

Then we have

$$S_2 = Z\Pi_U^{\perp}\Phi^T(\Phi\Pi_U^{\perp}\Phi^T)^{-1}\Phi\Pi_U^{\perp}Z^T. \quad \cdots\cdots (36)$$

Using the result of (27), the equation (34) can be rewritten as

$$S_2 = \widetilde{R}_{22}\widetilde{R}_{32}^T\Lambda^{-1}\widetilde{R}_{32}\widetilde{R}_{22}^T \quad \cdots\cdots (37)$$

where $\Lambda = \widetilde{R}_{32}\widetilde{R}_{32}^T + \widetilde{R}_{33}\widetilde{R}_{33}^T$. From the results of (28) and (37), we see that the estimate of the extended observability matrix is yielded by computing SVD of S_2 in (36). Furthermore the S_2 converges, that is,

$$\lim_{N\to\infty}\frac{1}{N^2}S_2 = \Gamma_i \frac{1}{N} X \Pi_U^{\perp} \Phi^T (\Phi \Pi_U^{\perp} \Phi^T)^{-1} \times \frac{1}{N} \Phi \Pi_U^{\perp} X^T \Gamma_i^T. \quad (38)$$

From the above cases, we see that the same framework can be applied by replacing the Hankel matrix Y consisted of the perturbed output by the matrix $\widehat{Z}$ including the instrumental variable.

4. Recursive computation

In this section, the practical implementation for the proposed method is given. We will show a relationship between the least squares residual and the matrix yielded by the Schur complement, and derive a recursive formula for an error covariance matrix in the 4SID method.

4.1 Noise-free case We have obtained the following relation

$$Y \Pi_U^{\perp} Y^T = R_{22} R_{22}^T,$$

therefore we consider a recursive algorithm of the left hand side of (21). The matrix $Y\Pi_U^{\perp}Y^T$ can be rewritten as

$$Y \Pi_U^{\perp} Y^T = Y \Pi_U^{\perp} (Y \Pi_U^{\perp})^T \quad (39)$$

We consider the matrix as follows

$$Y \Pi_U^{\perp} = Y - Y U^T (U U^T)^{-1} U = Y - \widehat{G}_N U \quad (40)$$

where $\widehat{G}_N$ is defined as

$$\widehat{G}_N := Y U^T (U U^T)^{-1}, \quad (41)$$

and we denote an error $Y - \widehat{G}_N U$ by $\widehat{E}$.

We can regard the matrix $\widehat{E}$ as a least squares residual, then $Y\Pi_U^{\perp}Y^T$ a squared sum of residuals can be denoted by an error covariance matrix $\widehat{E}\widehat{E}^T$. The normal equation can be represented by

$$(Y - \widehat{G}_N U) U^T = 0 \quad (42)$$

Equation (39) can be rewritten as

$$\widehat{E}\widehat{E}^T = (Y - \widehat{G}_N U)(Y - \widehat{G}_N U)^T = Y Y^T - \widehat{G}_N U Y^T \quad (43)$$

From the results of (42) and (43), the extended normal equation can be represented by

$$\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix} \begin{bmatrix} U U^T & U Y^T \\ Y U^T & Y Y^T \end{bmatrix} = \begin{bmatrix} O & \widehat{E}_N \widehat{E}_N^T \end{bmatrix} \quad (44)$$

The submatrix of the Hankel matrix $U_{k,i,N+1}$ is defined as

$$u_i(k+N) := \begin{bmatrix} u_{k+N}^T & u_{k+N+1}^T & \cdots & u_{k+N+i-1}^T \end{bmatrix}^T \quad (45)$$

and y_i is defined in a similar way. Then the Hankel matrices $U_{k,i,N+1}$ and $Y_{k,i,N+1}$ are partitioned such as

$$U_{k,i,N+1} = \begin{bmatrix} U_{k,i,N} & | & u_i(k+N) \end{bmatrix} \quad (46)$$

$$Y_{k,i,N+1} = \begin{bmatrix} Y_{k,i,N} & | & y_i(k+N) \end{bmatrix} \quad (47)$$

For brevity we denote $U_{k,i,N}$, $Y_{k,i,N}$ by U_N, Y_N, and $u_i(k+N)$, $y_i(k+N)$ by u_i, y_i, respectively. We consider a matrix consisted of the Hankel matrix U_{N+1} and Y_{N+1}. The following relation is obtained from (47).

$$\begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} = \begin{bmatrix} U_N \\ Y_N \end{bmatrix} \begin{bmatrix} U_N^T & Y_N^T \end{bmatrix} + \begin{bmatrix} u_i \\ y_i \end{bmatrix} \begin{bmatrix} u_i^T & y_i^T \end{bmatrix} \quad (48)$$

Equation (48) is multiplied by a matrix $\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix}$, and then the following is obtained

$$\begin{bmatrix} -\widehat{G}_N & I \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} = \begin{bmatrix} O & \widehat{E}_N \widehat{E}_N^T \end{bmatrix} + e_i \begin{bmatrix} u_i^T & y_i^T \end{bmatrix} \quad (49)$$

where

$$e_i := y_i - \widehat{G}_N u_i \quad (50)$$

We introduce the vectors $\boldsymbol{k}_{N+1}$ and $\widehat{y}_i(k+N)$ which satisfy the following equation;

$$\begin{bmatrix} \boldsymbol{k}_{N+1}^T & O \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} = \begin{bmatrix} u_i^T & \widehat{y}_i(k+N)^T \end{bmatrix} \quad (51)$$

Equations (49) and (51) multiplied by e_i give the following:

$$\begin{bmatrix} -\widehat{G}_N - e_i \boldsymbol{k}_{N+1}^T & I \end{bmatrix} \begin{bmatrix} U_{N+1} \\ Y_{N+1} \end{bmatrix} \times \begin{bmatrix} U_{N+1}^T & Y_{N+1}^T \end{bmatrix} = \begin{bmatrix} O & \widehat{E}_N \widehat{E}_N^T \end{bmatrix} + \begin{bmatrix} O & e_i \{y_i - \widehat{y}_i(k+N)\}^T \end{bmatrix} \quad (52)$$

Comparing equation (52) with (44), an error covariance $\widehat{E}_{N+1}\widehat{E}_{N+1}^T$ and estimates $\widehat{G}_{N+1}$ are obtained as

$$\widehat{G}_{N+1} = \widehat{G}_N + e_i \boldsymbol{k}_{N+1}^T \quad (53)$$

$$\widehat{E}_{N+1}\widehat{E}_{N+1}^T = \widehat{E}_N \widehat{E}_N^T + e_i \{y_i - \widehat{y}_i(k+N)\}^T \quad (54)$$

From equation (51), k_{N+1} and $\widehat{y}_i(k+N)$ are obtained as follows

$$\boldsymbol{k}_{N+1}^T = u_i^T P_{N+1} \quad (55)$$

$$\widehat{y}_i(k+N) = \widehat{G}_{N+1} u_i \quad (56)$$

where the matrix P_{N+1} is defined as

$$P_{N+1} := (U_{N+1} U_{N+1}^T)^{-1}. \quad (57)$$

Using the matrix inversion lemma, P_{N+1} can be represented by

$$P_{N+1} = P_N - P_N u_i u_i^T P_N / \alpha \quad (58)$$

where α is defined by

$$\alpha := 1 + u_i^T P_N u_i. \quad (59)$$

Using equations (55) and (58), the relation between $\widehat{G}_N$ and $\widehat{G}_{N+1}$ can be represented by

$$\widehat{G}_{N+1} = \widehat{G}_N + e_i u_i^T P_N/\alpha. \quad \cdots\cdots\cdots\cdots\cdots\cdots (60)$$

Therefore, a recursive formula is summarized as follows;

$$\begin{aligned} \alpha &= 1 + u_i^T P_N u_i \\ e_i &= y_i - \widehat{G}_N u_i \\ \widehat{G}_{N+1} &= \widehat{G}_N + e_i u_i^T P_N/\alpha \\ \widehat{E}_{N+1}\widehat{E}_{N+1}^T &= \widehat{E}_N\widehat{E}_N^T + e_i e_i^T/\alpha \\ P_{N+1} &= P_N - P_N u_i u_i^T P_N/\alpha \\ \mathcal{R}_{N+1} &= \mathcal{R}_N + \frac{e_i e_i^T}{1 + u_i^T P_N u_i} \quad \cdots\cdots\cdots (61) \end{aligned}$$

where the error covariance $\mathcal{R}_N := \widehat{E}_N\widehat{E}_N^T$. Using the matrix obtained by QR factorization in (9), the matrix $\widehat{G}_N$ can be rewritten as follows.

$$\begin{aligned} \widehat{G}_N &= YU^T(UU^T)^{-1} \\ &= R_{21}R_{11}^{-1} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (62) \end{aligned}$$

A recursive algorithm is described as follows.**step1** Store the initial value of the matrix P_N, $\widehat{G}_N$ and $\mathcal{R}_N$.
step2 If the new data sets u_i and y_i are available, update the matrix $\mathcal{R}_{N+1}$ and the required values by the recursive formula (61).
step3 Compute SVD of the matrix $\mathcal{R}_{N+1}$ as proposed in equation (10).
step4 Using the matrix E_n and $E_n^\perp$ given in step3, compute the quadruple of the system matrices $[A,\ B,\ C,\ D]$ in a similar way in **MOESP algorithm**, except for equation (14) which is rewritten such as:

$$[\ \xi_1 \quad \xi_2 \quad \cdots \quad \xi_i\ ] = (E_n^\perp)^T \widehat{G}_{N+1} \quad \cdots\cdots\cdots (63)$$

step5 If the next data sets are given, return to the step2.

4.2 **Noisy case** We have obtained the following result

$$\begin{aligned} \widehat{Z}\Pi_U^\perp \widehat{Z}^T &= \widehat{Z}\Pi_U^\perp(\widehat{Z}\Pi_U^\perp)^T \\ &= \widetilde{R}_{22}\widetilde{R}_{32}^T\Lambda^{-1}\widetilde{R}_{32}\widetilde{R}_{22}^T, \quad \cdots\cdots\cdots\cdots (64) \end{aligned}$$

then we consider a recursive formula of the matix $\widehat{Z}\Pi_U^\perp\widehat{Z}^T$. The matrix $\widehat{Z}\Pi_U^\perp$ can be rewritten as

$$\begin{aligned} \widehat{Z}\Pi_U^\perp &= Z\Pi_\Omega\Pi_U^\perp \\ &= \widehat{Z} - \widehat{G}_N U \\ &= \widehat{G}_N^*\Omega - \widehat{G}_N U \\ &= \widehat{E}_N - \widehat{E}_N^* \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (65) \end{aligned}$$

where

$$\widehat{G}_N^* := Y\Omega^T(\Omega\Omega^T)^{-1} \quad \cdots\cdots\cdots\cdots\cdots\cdots (66)$$

$$\widehat{E}_N^* := Y - \widehat{G}_N^*\Omega \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (67)$$

We denote $\widehat{Z}\Pi_U^\perp$ by $\widetilde{E}_N$, then $\widehat{Z}\Pi_U^\perp\widehat{Z}^T$ is represented by $\widetilde{E}_N\widetilde{E}_N^T$. From equation (65), the following equation is obtained.

$$\begin{aligned} \widetilde{E}_N\widetilde{E}_N^T &= (\widehat{E}_N - \widehat{E}_N^*)(\widehat{E}_N - \widehat{E}_N^*)^T \\ &= \widehat{E}_N\widehat{E}_N^T - \widehat{E}_N^*\widehat{E}_N^{*T} \quad \cdots\cdots\cdots\cdots\cdots (68) \end{aligned}$$

Therefore we see that $\widetilde{E}_N\widetilde{E}_N^T$ is yielded by computing $\widehat{E}_N\widehat{E}_N^T$ and $\widehat{E}_N^*\widehat{E}_N^{*T}$. The matrix Ω is defined as

$$\Omega := [\omega_i(k) \quad \omega_i(k+1) \quad \cdots \quad \omega_i(k+N-1)] \quad (69)$$

where

$$\omega_i(k) := \left[u_i(k)^T \quad \phi_k^T \quad \phi_{k+1}^T \quad \cdots \quad \phi_{k+i-1}^T\right]^T \quad (70)$$

and ϕ is an element of the instrumental variable Φ. $\widehat{E}_N^*$, $\widehat{G}_N^*$ and Ω are corresponded $\widehat{E}_N$, $\widehat{G}_N$ and U, then a recursive formula of the error covariance matrix $\widehat{E}_N^*\widehat{E}_N^{*T}$ is summarized as follows;

$$\begin{aligned} \beta &= 1 + \omega_i^T Q_N \omega_i \\ e_i^* &= y_i - \widehat{G}_N^*\omega_i \\ \widehat{G}_{N+1}^* &= \widehat{G}_N^* + e_i^*\omega_i^T Q_N/\beta \\ \widehat{E}_{N+1}^*\widehat{E}_{N+1}^{*T} &= \widehat{E}_N^*\widehat{E}_N^{*T} + e_i^* e_i^{*T}/\beta \\ Q_{N+1} &= Q_N - Q_N\omega_i\omega_i^T Q_N/\beta \\ \mathcal{R}_{N+1}^* &= \mathcal{R}_N^* + \frac{e_i^* e_i^{*T}}{1 + \omega_i^T Q_N \omega_i} \quad \cdots\cdots\cdots (71) \end{aligned}$$

where $\mathcal{R}_N^* := \widehat{E}_N^*\widehat{E}_N^{*T}$, $Q_N := (\Omega_N\Omega_N^T)^{-1}$. Therefore a recursive formula of $\widetilde{E}\widetilde{E}^T$ is obtained

$$\widetilde{\mathcal{R}}_{N+1} = \mathcal{R}_{N+1} - \mathcal{R}_{N+1}^* \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (72)$$

where $\widetilde{\mathcal{R}}_N := \widetilde{E}_N\widetilde{E}_N^T$.

A recursive algorithm for the noisy case is described as follows.**step1**$'$ Store the initial value of the matrix P_N, $\widehat{G}_N$, $\mathcal{R}_N$, Q_N, $\widehat{G}_N^*$ and $\mathcal{R}_N^*$
step2$'$ If the new data sets u_i and y_i are available, update the matrix $\widetilde{\mathcal{R}}_{N+1}$ using $\mathcal{R}_{N+1}$, $\mathcal{R}_{N+1}^*$ and the required values by the recursive formula (71).
step3$'$ Compute SVD of the matrix $\widetilde{\mathcal{R}}_{N+1}$ as in equation (10).
step4$'$ Using the matrix E_n and $E_n^\perp$ given in step3, compute the quadruple of the system matrices $[A,\ B,\ C,\ D]$ in a similar way in recursive algorithm for noise-free case.
step5$'$ If the next data sets are given, return to the step2.

4.3 **Numerical example** In this section, we applied the recursive algorithm presented in this paper to identify the following discrete time linear system;

$$x_{k+1} = \begin{bmatrix} 0.8 & -0.4 & 0.2 \\ 0 & 0.3 & -0.5 \\ 0 & 0 & 0.5 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 0 & -0.6 \\ 0.5 & 0 \end{bmatrix} u_k \quad (73)$$

$$y_k = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.05 \\ 0.02 \end{bmatrix} v_k \quad \cdots\ (74)$$

where u_k is constructed by 2 inputs, y_k by 2 outputs and x_k is 3 state vector, respectively. The experiment was conducted with MATLAB package. In order to use our recursive procedure, we generated 7500 samples of input-output data and took the input u_k equal to a zero-mean white noise of unit variance, the noise v_k is as in a similar way. We used the first 50 samples to produce an estimate as initial value in off-line. Using the recursive algorithm for noisy case, we estimated a sequence of state space models. The instrumental variable is selected as

$$\Phi = U_{k-i,i,N} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (75)$$

and an auxiliary order $i = 10$. We used principal angles [5] as the indicator of the similarity between the column space of Γ_i and the estimate E_n. Fig.1 shows that the estimate E_n asymptotically tend to the true values for increasing step number.

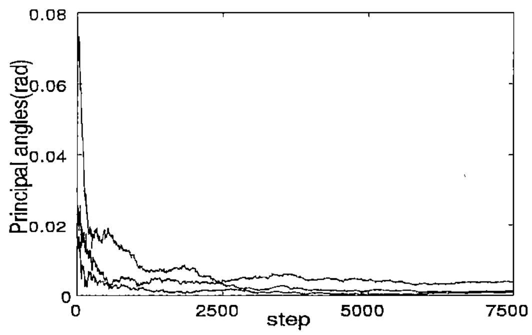

Fig. 1. Principal angles

5. Conclusion

In this paper, we have presented another interpretation of the 4SID method by subspace extraction via Schur complement, and showed that the estimate of the extended observability matrix is obtained from Schur complement matrix consisted of the Hankel matrices of input-output data. It has been shown that the proposed procedure can be applicable to both cases, MOESP and IV-based one. We proposed a recursive formula for the error covariance matrix in the 4SID method, and the algorithm with an instrumental variable has been also developed. The presented results are useful to analyze the properties of subspace-based identification methods.

(Manuscript received june 2, 2000, revised Nov. 2, 2000)

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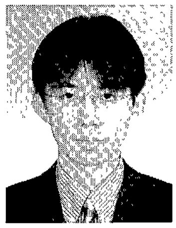

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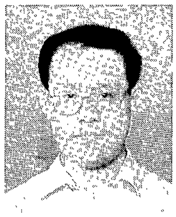

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