Paper

Study on Lyapunov Functions for Liénard-type Nonlinear Systems

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In this paper, a generalized Lyapunov function for the Liénard-type nonlinear system which is important as a representative system expressing LRC electric circuits and mechanical spring-mass systems etc., is constructed using the Lagrange-Charpit method. The Lyapunov function includes particular nonlinear terms as arbitrary functions, by which the quadratic term appearing in the Luré-type Lyapunov function can be extended. The result yields all the conventional Lyapunov functions as special cases, changing the forms of the arbitrary functions. To investigate the relation between the arbitrary function in the Lyapunov function and the stability region obtained, the stability boundaries for various types of the arbitrary functions are illustrated in the application to a simple system. In addition, numerical values of the time derivative of the Lyapunov function along the stability boundary are calculated to study the relation between the stability region and the values of the time derivative.

Keywords: Liénard-type Nonlinear Systems, Lagrange-Charpit method Stability, Lyapunov function.

1. Introduction

It is well known that the nonlinear system given by Liénard's equation is important as a general system expressing LRC electric circuits, spring-mass systems and electric machinery. In the stability analysis of the system, it is common to utilize the Lyapunov method, and the energy-type Lyapunov function is widely used. The Lyapunov method has two main uses, i.e., establishment of the stability of a null solution of the system and determination of a stability region for the system. The latter is often important to system engineers, because a lot of systems appearing in engineering have nonlinearities in which only local stability is discussed. Moreover, the Lyapunov method gives only sufficient conditions for obtaining the stability. Hence, a Lyapunov function which gives good approximation to the true stability boundary is desired. Thus, several methods (1) – (8) for constructing Lyapunov functions have been applied to the nonlinear systems, aimed at obtaining better stability estimations.

Yu et al. (1) and DeSarkar et al. (2) have applied the Zubov method to a synchronous machine system represented by Liénard's equation. Although an approximate solution with the truncated series form is used in the method, a higher-order approximation does not necessarily give a larger stability boundary. Also the scalar function, as a clue to the solution of Zubov's partial differential equation, can not be easily established. For the same system, another Lyapunov function has been presented by Prabhakara et al. (3), using the generalized Zubov method. In that method, however, the solution of any stability problem depends on the transformation of the variables, while suitable transformation forms are not always available. Miyagi et al. (7) presented the way of generalizing Lyapunov function, introducing the particular term which contributes to the extension of the stability region. Stability theorems were used to derive the Lyapunov function of the Liénard's equation with nonlinear damping. This was followed by the improved Lyapunov function using a generalized energy function method (8). These papers, however, dealt with the simple system given by Liénard's equation. Liénard-type nonlinear system given by a system of n second-order differential equations, a generalization of Liénard's equation, has not been thoroughly discussed. Stability of that type of system has been studied to some level in the conventional literatures (9) – (14).

This paper presents a generalized Lyapunov function for the Liénard- type nonlinear system, using the Lagrange-Charpit method (4) (5) which is a well-known technique for solving partial differential equations. The stability of Liénard-type nonlinear systems has been discussed by Miyagi et al. (10), and this paper deals with the system categorized as stable in that paper. First, a generalized Lyapunov function for single Liénard's equation is constructed. The function includes an arbitrary function, by which the quadratic term appearing in the Luré-type Lyapunov function (6) is extended. All Lyapunov functions presented so far are obtainable by means of changing the form of the arbitrary function. As the stability region obtained by the Lyapunov function varies with the shape of the arbitrary function, we investigate the relation between the arbitrary function and the stability region obtained. Further, we investigate the relation between the obtained stability region and the values of the time derivative of the Lyapunov function along the stability boundary. After these investigations, we construct the generalized Lyapunov function for the Liénard-type nonlinear system given by a system of n second-order differential equations. The Lyapunov function includes extended terms which contribute to the extension of the obtained stability boundary.

2. Liénard-type nonlinear system

The Liénard-type nonlinear system considered here is of the form [10]

$$\ddot{\mathbf{y}} + \mathbf{G}(\mathbf{y})\dot{\mathbf{y}} + \gamma(\mathbf{y}) = \mathbf{0} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where $\mathbf{y}$ is an n-dimensional vector, $\mathbf{G}(\mathbf{y})(> \mathbf{0})$ is a nonlinear damping defined by

$$\begin{aligned}
&\mathbf{G}(\mathbf{y}) \\
&= \mathbf{D}\begin{bmatrix} g_1(\sigma_1) & & & 0 \\ & g_2(\sigma_2) & & \\ & & \ddots & \\ 0 & & & g_m(\sigma_m) \end{bmatrix}\mathbf{B}^{\mathbf{T}} \\
&= \mathbf{D}\{\mathrm{diag}\,[g_i(\sigma_i)]\}\,\mathbf{B}^T
\end{aligned}$$

$\mathbf{D}$ and $\mathbf{B}$ are $n \times m$ matrices, $\gamma(\mathbf{y}) = \mathbf{B}\mathbf{f}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} = \mathbf{B}^T\mathbf{y}$ and $\mathbf{f}^T(\boldsymbol{\sigma}) = [f_1(\sigma_1), f_2(\sigma_2), \cdots, f_m(\sigma_m)]$.

The nonlinear functions $g_i(\sigma_i)$ and $f_i(\sigma_i)$ are assumed to be continuous, differentiable and to satisfy the following conditions:

I. $g_i(\sigma_i) > 0$ for $\sigma_i \neq 0$
II. $\sigma_i f_i(\sigma_i) > 0$ for $\sigma_i \neq 0$, $\quad f(0) = 0$
III. $|\Psi_i(\sigma_i)| \to \infty$ as $|\sigma_i| \to \infty$

where $\Psi_i(\sigma_i) = \int_0^{\sigma_i} g_i(\sigma_i)d\sigma_i$

Making the change of variables: $\mathbf{y} = \mathbf{x}_1$, $\dot{\mathbf{y}} = \mathbf{x}_2$, we can rewrite system (1) in the form of first-order simultaneous equations as

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}); \quad (\mathbf{h}(\mathbf{0}) = \mathbf{0}) \quad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

where $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$, $\mathbf{h}(\mathbf{x}) = [\mathbf{h}_1^T, \mathbf{h}_2^T]^T$, $\mathbf{x}_i = [x_{i1}, x_{i2}, \cdots, x_{in}]^T$, $\mathbf{h}_i = [h_{i1}, h_{i2}, \cdots, h_{in}]^T$, $\quad i = 1, 2$

3. Lagrange-Charpit method [4]

The problem concerning the stability analysis of the equilibrium state of the system (2) is formally taken up by the search for a Lyapunov function $V = V(\mathbf{x})$ which satisfies the partial differential equation

$$F(\mathbf{x}, V, \mathbf{P}) = \mathbf{P}^T\mathbf{h}(\mathbf{x}) + \psi(\mathbf{x}) = \mathbf{0} \quad \cdots\cdots\cdots (3)$$

where $\mathbf{P} = \frac{\partial V}{\partial \mathbf{x}} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^T$, $\mathbf{P}_i = [P_{i1}, P_{i2}, \cdots, P_{in}]^T$ and $\psi(\mathbf{x})$ is an arbitrary non-negative function whose opposite sign may be the time derivative of the obtained Lyapunov function. Separation of $\mathbf{P}$ into $\mathbf{P}_1$, $\mathbf{P}_2$ simplifies the application of the Lagrange-Charpit method to second-order differential equations.

Then we determine the scalar functions V and ψ. The characteristic equation for (3) is given by

$$\begin{aligned}
&\frac{dx_{11}}{\frac{\partial F}{\partial P_{11}}} = \cdots = \frac{dx_{1n}}{\frac{\partial F}{\partial P_{1n}}} = \frac{dx_{21}}{\frac{\partial F}{\partial P_{21}}} = \cdots = \frac{dx_{2n}}{\frac{\partial F}{\partial P_{2n}}} \\
&= \frac{-dP_{11}}{\frac{\partial F}{\partial x_{11}} + P_{11}\frac{\partial F}{\partial V}} = \cdots = \frac{-dP_{1n}}{\frac{\partial F}{\partial x_{1n}} + P_{1n}\frac{\partial F}{\partial V}} \\
&= \frac{-dP_{21}}{\frac{\partial F}{\partial x_{21}} + P_{21}\frac{\partial F}{\partial V}} = \cdots = \frac{-dP_{2n}}{\frac{\partial F}{\partial x_{2n}} + P_{2n}\frac{\partial F}{\partial V}} \quad \cdot (4)
\end{aligned}$$

where $\frac{\partial F}{\partial x_{11}}, \cdots, \frac{\partial F}{\partial x_{1n}}, \frac{\partial F}{\partial x_{21}}, \cdots, \frac{\partial F}{\partial x_{2n}}$ include $\frac{\partial \psi}{\partial x_{11}}, \cdots, \frac{\partial \psi}{\partial x_{1n}}, \frac{\partial \psi}{\partial x_{21}}, \cdots, \frac{\partial \psi}{\partial x_{2n}}$, respectively.

Next, according to Lagrange-Charpit method, we derive $2n-1$ equations of $\mathbf{P}$ from (4) or the related equations given by multiplying both numerator and denominator of (4) by some constants or variables.

Thus, if we can obtain $2n-1$ equations:

$$\begin{aligned}
Z_1\left(\mathbf{x}, V, \mathbf{P}, \frac{\partial \psi}{\partial \mathbf{x}}\right) &= 0 \\
Z_2\left(\mathbf{x}, V, \mathbf{P}, \frac{\partial \psi}{\partial \mathbf{x}}\right) &= 0 \\
&\cdots \\
Z_{2n-1}\left(\mathbf{x}, V, \mathbf{P}, \frac{\partial \psi}{\partial \mathbf{x}}\right) &= 0
\end{aligned} \quad \cdots\cdots\cdots\cdots\cdots\cdots (5)$$

containing at least one element of the variable $\mathbf{P}$, then the unknown functions $\frac{\partial \psi}{\partial x_{11}}, \cdots, \frac{\partial \psi}{\partial x_{1n}}, \frac{\partial \psi}{\partial x_{21}}, \cdots, \frac{\partial \psi}{\partial x_{2n}}$ and ψ are determined from the following conditions:

$$\begin{aligned}
[Z_i, Z_j] &= \sum_{k=1}^{2}\left(\left[\frac{dZ_i}{dx_k}\right]^T \frac{\partial Z_j}{\partial P_k} - \left[\frac{dZ_j}{dx_k}\right]^T \frac{\partial Z_i}{\partial P_k}\right) \\
&= 0 \\
[Z_i, F] &= \sum_{k=1}^{2}\left(\left[\frac{dZ_i}{dx_k}\right]^T \frac{\partial F}{\partial P_k} - \left[\frac{dF}{dx_k}\right]^T \frac{\partial Z_i}{\partial P_k}\right) \\
&= 0
\end{aligned} \quad \cdots\cdots\cdots\cdots\cdots\cdots (6)$$

where $i, j = 1, 2, \cdots, 2n-1, i \neq j$ and

$$\frac{dZ_i}{dx_k} = \frac{\partial Z_i}{\partial x_k} + P_k\frac{\partial Z_i}{\partial V} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

Equation (6) is Jacobi's brackets [15] which imply the necessary and sufficient conditions for both the cases that functions $Z_1, Z_2, \ldots, Z_{2n-1}$ and F have a common solution, and the Pfaffian differential equation is integrable independent of the path of integration. In this stage, unknown functions $\frac{\partial \psi}{\partial x_{11}}, \cdots, \frac{\partial \psi}{\partial x_{1n}}, \frac{\partial \psi}{\partial x_{21}}, \cdots, \frac{\partial \psi}{\partial x_{2n}}$ and ψ can be expressed using the variable $\mathbf{x}$.

If (5) can be solved so as to give $\mathbf{P}$ as a function of $\mathbf{x}$ and V, such that

$$\mathbf{P} = \mathbf{P}(\mathbf{x}, V) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

the Pfaffian differential equation

$$\mathbf{P}^T d\mathbf{x} = (\nabla V) d\mathbf{x} \quad \cdots\cdots (9)$$

is integrable, because of the fact that $\mathbf{P}$ satisfies (6). Hence, integrating (9) , a possible Lyapunov function V is given as

$$V(\mathbf{x}) = \int_0^x \mathbf{P}^T d\mathbf{x}, \quad \cdots\cdots (10)$$

with

$$\dot{V}(\mathbf{x}) = -\psi(\mathbf{x}) \quad \cdots\cdots (11)$$

When V in (10) satisfies the following Lyapunov criteria:

(i) $V(\mathbf{x})$ is a continuous scalar function which has continuous first partial derivatives with respect to $\mathbf{x}$.
(ii) $V(\mathbf{x}) = 0$ for $\mathbf{x} = 0$
(iii) $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$
(iv) $\dot{V}(\mathbf{x}) \leq 0$
(v) $\dot{V}(\mathbf{x})$ is not identically equal to zero along any trajectory of the system other than the origin.

in a region R of $\mathbf{x}$ space, V is a Lyapunov function, and system (2) is verified to be asymptotically stable in the neighbourhood of the origin. In this construction procedure, the integration constants appearing when integrating (4) and (9) may be neglected so that $V(\mathbf{x})$ satisfies Lyapunov's criteria.

4. Lyapunov function for Liénard's equation

In this section, a generalized Lyapunov function for Liénard's equation is constructed, using the Lagrange-Charpit method described in the previous section.

4.1 Generalized Lyapunov function for Liénard's equation

Let us consider the Liénard's equation ($n = 1$):

$$\ddot{y} + g(y)\dot{y} + f(y) = 0 \quad \cdots\cdots (12)$$

where y is the scalar unknown function. Making the change of variables $y = x_1$, $\dot{y} = x_2$, we can rewrite system (12) in the form of (2) as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -g(x_1)x_2 - f(x_1) \end{bmatrix} \quad \cdots\cdots (13)$$

Then, using the Lagrange-Charpit method, the Lyapunov function is obtained as a function which satisfies the linear partial differential equation (3). The characteristic equation becomes

$$\begin{aligned} \frac{dx_1}{x_2} &= \frac{dx_2}{-g(x_1)x_2 - f(x_1)} \\ &= \frac{dP_1}{\{g'(x_1)x_2 + f'(x_1)\}P_2 - \frac{\partial\psi}{\partial x_1}} \\ &= \frac{dP_2}{g(x_1)P_2 - P_1 - \frac{\partial\psi}{\partial x_2}} \end{aligned} \quad \cdots\cdots (14)$$

where $f'(x_1) = \frac{df(x_1)}{dx_1}$ and $g'(x_1) = \frac{dg(x_1)}{dx_1}$

From (14) , two equations can be derived containing P_1, P_2 and $\frac{\partial\psi}{\partial x_2}$ as

$$\begin{aligned} Z_1 &= \alpha\phi(x_1) + \beta x_2 - P_2 = 0 \\ Z_2 &= \alpha\phi'(x_1)x_2 - \beta\{g(x_1)x_2 + f(x_1)\} \\ &\quad -g(x_1)P_2 + P_1 + \frac{\partial\psi}{\partial x_2} = 0 \end{aligned} \quad \cdots\cdots (15)$$

where α and β are arbitrary real constants, $\phi(x_1)$ is an arbitrary function and $\phi'(x_1) = \frac{d\phi(x_1)}{dx_1}$. An emphasis of this paper is the introduction of the arbitrary nonlinear function ϕ.

Applying (6) to Z_1, Z_2 and F gives

$$\begin{aligned} [Z_1, Z_2] &= 2\{\beta g(x_1) - \alpha\phi'(x_1)\} - \frac{\partial^2\psi}{\partial x_2{}^2} = 0 \\ [Z_2, F] &= \{\alpha\phi''(x_1) - \beta g'(x_1)\}x_2^2 \\ &\quad + \frac{\partial^2\psi}{\partial x_1 \partial x_2}x_2 - \frac{\partial\psi}{\partial x_1} \\ &\quad + \alpha\{\phi(x_1)f'(x_1) + \phi'(x_1)f(x_1)\} = 0 \end{aligned} \quad \cdots\cdots (16)$$

where $\phi''(x_1) = \frac{d\phi'(x_1)}{dx_1}$. The condition $[Z_1, F] = 0$ becomes identical with $Z_2 = 0$.

The unknown functions $\frac{\partial\psi}{\partial x_2}$ and ψ are determined from (16). As the results, we have

$$\begin{aligned} \frac{\partial\psi}{\partial x_2} &= 2\{\beta g(x_1) - \alpha\phi'(x_1)\}x_2 + \Phi(x_1) \\ \psi &= \{\beta g(x_1) - \alpha\phi'(x_1)\}x_2^2 \\ &\quad + \Phi(x_1)x_2 + \alpha\phi(x_1)f(x_1) \end{aligned} \quad \cdots\cdots (17)$$

where Φ is an arbitrary function. Inequalities

$$\begin{aligned} &\beta g(x_1) - \alpha\phi'(x_1) \geq 0, \\ &\alpha\phi(x_1)f(x_1) > 0, \quad x_1 \neq 0 \end{aligned} \quad \cdots\cdots (18)$$

must be satisfied. Then we choose Φ as

$$\Phi(x_1) = -2\sqrt{K(x_1)\alpha\phi(x_1)f(x_1)} \quad \cdots\cdots (19)$$

in order for ψ to be a perfect square form written by

$$\psi = \left[\sqrt{K(x_1)}x_2 - \sqrt{\alpha\phi(x_1)f(x_1)}\right]^2 \quad \cdots\cdots (20)$$

where $K(x_1) = \beta g(x_1) - \alpha\phi'(x_1) \geq 0$.

Solving (15) for P_1 and P_2, we obtain

$$
\begin{aligned}
P_1 &= \alpha\phi(x_1)g(x_1)+\beta f(x_1)+\alpha\phi'(x_2)\\
&\quad +2\sqrt{K(x_1)\alpha\phi(x_1)f(x_1)}\\
P_2 &= \alpha\phi(x_1)+\beta x_2
\end{aligned}
\quad\cdots\cdots (21)
$$

Here, replacements $\alpha = 1$ and $\beta = 1$ will be of help in inspecting the definiteness of the resultant V. Thus the scalar function V in (10) is given as

$$
\begin{aligned}
V &= \frac{1}{2}\{x_2+\phi(x_1)\}^2+\int_0^{x_1} f(x_1)dx_1\\
&\quad +\int_0^{x_1}\phi(x_1)K(x_1)dx_1\\
&\quad +2\int_0^{x_1}\sqrt{K(x_1)\phi(x_1)f(x_1)}dx_1 \quad\cdots\cdots (22)
\end{aligned}
$$

with

$$\dot{V} = -\psi = -[\sqrt{K(x_1)}x_2-\sqrt{\phi(x_1)f(x_1)}]^2 \cdot (23)$$

Now, the conditions in (18) result in

$$
\begin{aligned}
&\phi(0)=0, \quad x_1\phi(x_1)>0 \quad (x_1\neq 0)\\
&K(x_1)=g(x_1)-\phi'(x_1)\geq 0
\end{aligned}
\quad\cdots\cdots (24)
$$

Next we inspect the definiteness of V in the neighbourhood of the origin. According to Miyagi et al.[6], we rewrite the right-hand side of (22) except the first term, such that

$$V_0 = \int_0^{x_1} H(x_1)dx_1 \cdots\cdots (25)$$

where

$$
\begin{aligned}
H(x_1) &= \phi(x_1)K(x_1)+f(x_1)\\
&\quad +2\sqrt{K(x_1)\phi(x_1)f(x_1)} \cdots\cdots (26)
\end{aligned}
$$

Equation (26) can be rearranged in the forms

$$
H(x_1) = \begin{cases}
[\sqrt{K(x_1)\phi(x_1)} +\sqrt{f(x_1)}]^2, & x_1\geq 0\\
-[\sqrt{-K(x_1)\phi(x_1)} -\sqrt{-f(x_1)}]^2, & x_1<0
\end{cases}
\quad\cdots\cdots (27)
$$

Hence

$$x_1H(x_1)\geq 0 \cdots\cdots (28)$$

Then, V_0 is the positive function in the condition given by the inequality(28) . Therefore, V in (22) is a positive definite function, and satisfies conditions (i), (ii) and (iii). On the other hand, we can see that the system (13) has no singular points except the origin $\mathbf{x}=\mathbf{0}$. That implies $\dot{V}$ in (23) is not identically equal to zero along any system trajectory other than the origin. Thus, $\dot{V}$ satisfies conditions (iv) and (v). From the above discussions, V in (22) is a Lyapunov function, and the given system is verified to be asymptotically stable.

If we keep $\phi = 0(\equiv \phi_1)$, in the conditions given in (24), (22) results in

$$V = \frac{1}{2}x_2^2+\int_0^{x_1} f(x_1)dx_1 \equiv V_1 \cdots\cdots (29)$$

The above Lyapunov function is the well-known energy function. Next, $\phi = \int_0^{x_1} g(x_1)dx_1 \equiv \Gamma(x_1)(\equiv \phi_2)$ leads to

$$V = \frac{1}{2}\{x_2+\Gamma(x_1)\}^2+\int_0^{x_1} f(x_1)dx_1 \equiv V_2 \quad (30)$$

which is equivalent to that obtained by the generalized Zubov method[3] . Also, $\phi = \eta\Gamma(x_1)((\equiv \phi_3)(\eta$: constant, $0\leq\eta\leq 1)$ yields the result given in[4], that is

$$
\begin{aligned}
V &= \frac{1}{2}\{x_2+\eta\Gamma(x_1)\}^2+\int_0^{x_1} f(x_1)dx_1\\
&\quad +\frac{1}{2}\eta(1-\eta)\Gamma^2x_1\\
&\quad +2\sqrt{\eta(1-\eta)}\int_0^{x_1}\sqrt{\Gamma(x_1g(x_1)f(x_1)}dx_1\\
&\equiv V_3 \cdots\cdots (31)
\end{aligned}
$$

The generalized Lyapunov function derived in the literature[7] is also this type of function. The improved Lyapunov function derived by Kawamoto et al.[8] can be obtained as

$$
\begin{aligned}
V &= \frac{1}{2}\{x_2+\lambda(x_1)\Gamma(x_1)\}^2\\
&\quad +\int_0^{x_1} f(x_1)dx_1\\
&\quad +\int_0^{x_1}\lambda(x_1)\Gamma(x_1)K(x_1)dx_1\\
&\quad +2\int_0^{x_1}\sqrt{\lambda(x_1)\Gamma(x_1)K(x_1f(x_1)}dx_1\\
&\equiv V_4 \cdots\cdots (32)
\end{aligned}
$$

by setting $\phi = \lambda(x_1)\Gamma(x_1)(\equiv \phi_4)(\lambda(x_1)\geq 0)$.

In the above Lyapunov functions, $\phi_i(i=1,...,4)$ have been selected to satisfy the conditions given in (24) . Another ϕ apart from the above types of ϕ_i can be found as $\phi = g(x_1)\tanh x_1(\equiv \phi_5)$. This way, many types of Lyapunov functions will be derived by changing the form of ϕ. Hence, the Lyapunov function given in (22) is regarded as a generalized Lyapunov function for the system (12).

4.2 Arbitrary functions and asymptotic stability regions Under the conditions for $f_i(\sigma_i)$ and $g(\sigma_i)$, we may obtain a global stability result. For many applications, however, condition II for $f_i(\sigma_i)$ is not satisfied globally. Then, Lyapunov function V is used to estimate a region of asymptotic stability. In fact, the generalized function given in (22) displays its power in such cases. In order to show the superiority of the generalized Lyapunov function, let us consider a particular system given by $g(x_1) = D$ (D: a positive constant) and $f(x_1) = \sin(x_1 + \delta) - \sin\delta$, ($\delta$: a constant). For this case, there exists $x_1 f(x_1) \geq 0$ for $-\pi - 2\delta \leq x_1 \leq \pi - 2\delta$, and the system has asymptotic stability around the origin. First, in Fig. 1, we show some previously selected $\phi_i (i = 1, ..., 5)$ satisfying inequalities given in (24). The parameters are given by $D = 0.3$ and $\delta = 0.412$. Stability regions obtained from Lyapunov functions $V_i (i = 1, ..., 5)$ corresponding to ϕ_i respectively are shown in Fig.2, including the true stability region. From Figs.1 and 2, we can see that the obtained stability regions are fairly dependent on the shapes of the arbitrary functions ϕ_i. Fig.3 shows the numerical values of $\dot{V}$ along the boundaries of the various stability regions. Comparing these $x_1 - \dot{V}$ curves with the asymptotic stability regions shown in Fig.2, we may find that the value of $\dot{V}$ approaches zero as the stability boundary gets closer to the true stability boundary. This fact may present us a conjecture, i.e., "Lyapunov function V with $\dot{V} \equiv 0$, may give the integral surface of the given system". In other words, "if we can find the Lyapunov function V with $\dot{V} \equiv 0$, then the function gives the true stability region of the system". If our conjecture is correct, then one may try to find a suitable ϕ so that the value of $\dot{V}$ approaches zero. The means of determining the optimal ϕ is a topic of future discussion.

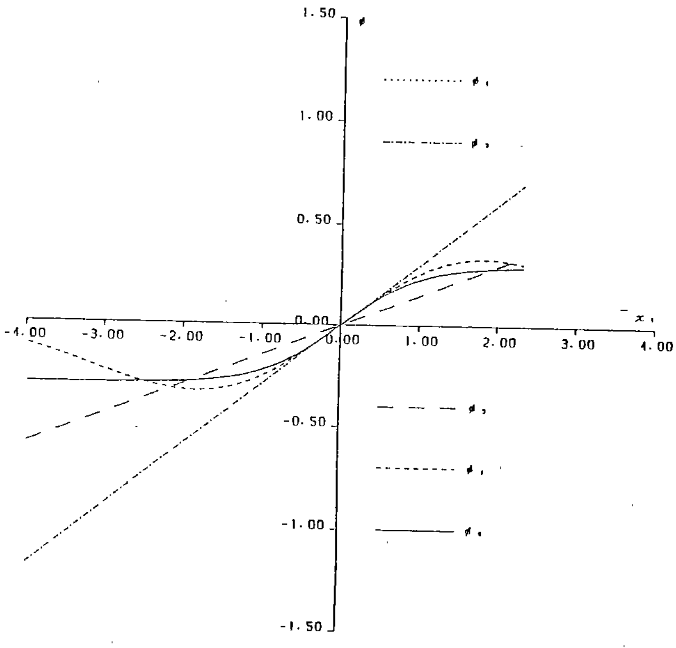

Fig. 1. Arbitrary functions $\phi(x_i)$

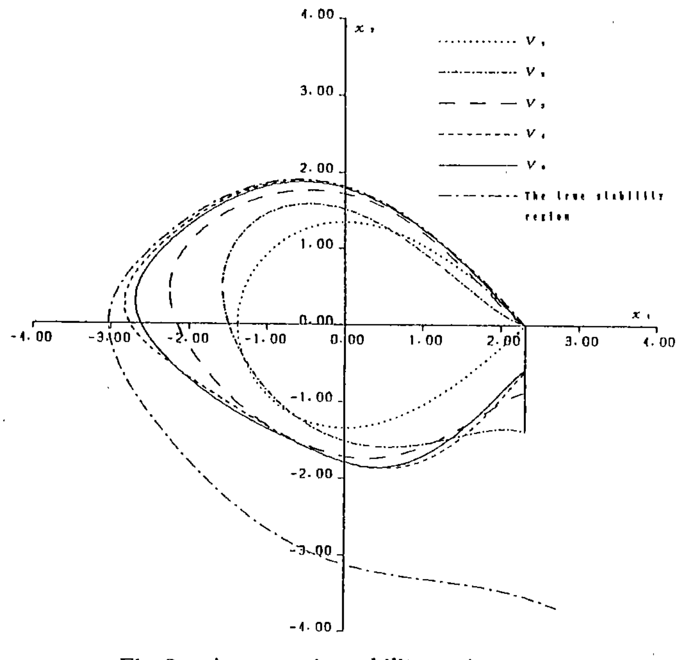

Fig. 2. Asymptotic stability regions

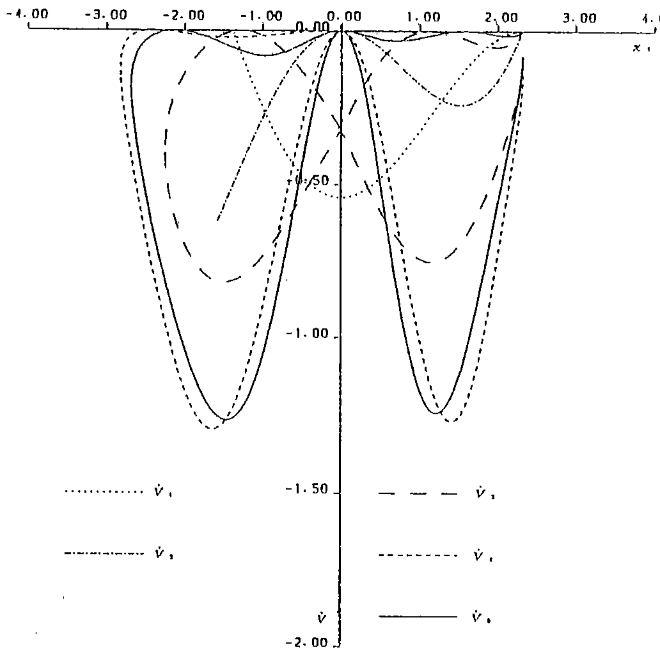

Fig. 3. Values of V on the asymptotic stability boundaries

5. Lyapunov function for Liénard-type nonlinear systems

In this section, we derive the generalized Lyapunov function for Liénard-type nonlinear systems. Stability of these types of systems has not been discussed so far, except by Miyagi et al. [9][10].

5.1 Generalized Lyapunov function for Liénard-type nonlinear systems Stability of Liénard-type nonlinear system (1), which is a generalization of Liénard's equation, has been studied for the case $\mathbf{D} = \mathbf{B}$ and $\mathbf{D}^T\mathbf{B} = \mathrm{diag}[\lambda_i]$ (λ_i: positive constants)[10]. In this section, we construct a generalized Lyapunov function which includes all Lyapunov functions presented so far, using the Lagrange-Charpit method.

The characteristic equation (4) with $\frac{\partial F}{\partial V}=0$ becomes;

$$\begin{aligned}
&\frac{dx_{11}}{h_{11}(\mathbf{x})}=\cdots=\frac{dx_{1n}}{h_{1n}(\mathbf{x})}\\
&=\frac{dx_{21}}{h_{21}(\mathbf{x})}=\cdots=\frac{dx_{2n}}{h_{2n}(\mathbf{x})}\\
&=\frac{-dP_{11}}{\frac{\partial F}{\partial x_{11}}}=\cdots=\frac{-dP_{1n}}{\frac{\partial F}{\partial x_{1n}}}\\
&=\frac{-dP_{21}}{\frac{\partial F}{\partial x_{21}}}=\cdots=\frac{-dP_{2n}}{\frac{\partial F}{\partial x_{2n}}}
\end{aligned} \quad \cdots\cdots\cdots (33)$$

From equation (33), $2n$ equations can be derived containing $\mathbf{P}_1, \mathbf{P}_2$ and $\frac{\partial \psi}{\partial \mathbf{x}_2}$ as

$$\begin{aligned}
&\mathbf{Z}^1=\alpha\mathbf{D}\{\text{ diag } [\phi_i(\sigma_i)]\}\mathbf{1}+\beta\mathbf{x}_2-\mathbf{P}_2=\mathbf{0}\\
&\mathbf{Z}^2=\alpha\mathbf{\Phi}'(\boldsymbol{\sigma})\mathbf{x}_2-\beta\{\mathbf{G}(\mathbf{x}_1)\mathbf{x}_2+\mathbf{B}\mathbf{f}(\boldsymbol{\sigma})\}\\
&\qquad+\mathbf{P}_1-\mathbf{G}(\mathbf{x}_1)\mathbf{P}_2+\frac{\partial\psi}{\partial\mathbf{x}_2}=\mathbf{0}
\end{aligned} \quad \cdots\cdots\cdots (34)$$

where $\mathbf{Z}^1=[Z_1, Z_2, \cdots, Z_n]^T$,
$\mathbf{Z}^2=[Z_{n+1}, Z_{n+2}, \cdots, Z_{2n}]^T$,
$\mathbf{\Phi}'(\boldsymbol{\sigma})=\mathbf{D}\{\text{ diag } [\phi_i'(\sigma_i)]\}\mathbf{B}^T$, $\phi_i'(\sigma_i)=\frac{d\phi_i(\sigma_i)}{\sigma_i}$.
α and β are arbitrary constants, $\phi_i(\sigma_i)$ are arbitrary functions and $\mathbf{1}$ is a vector with all components 1.

Applying (6) to $\mathbf{Z}^1, \mathbf{Z}^2$ and F gives

$$\begin{aligned}
&[\mathbf{Z}^1,\mathbf{Z}^2]=2\{\beta\mathbf{G}(\mathbf{x}_1)-\alpha\Phi'(\boldsymbol{\sigma})\}\\
&\qquad-\frac{\partial^2\psi}{\partial\mathbf{x}_2^2}=\mathbf{0}\\
&[\mathbf{Z}^2,F]=\begin{bmatrix}\mathbf{x}_2^T[\alpha\frac{\partial\Phi'(\boldsymbol{\sigma})}{\partial x_{11}}-\beta\frac{\partial\mathbf{G}(\mathbf{x}_1)}{\partial x_{11}}]\mathbf{x}_2\\ \mathbf{x}_2^T[\alpha\frac{\partial\Phi'(\boldsymbol{\sigma})}{\partial x_{12}}-\beta\frac{\partial\mathbf{G}(\mathbf{x}_1)}{\partial x_{12}}]\mathbf{x}_2\\ \vdots\\ \mathbf{x}_2^T[\alpha\frac{\partial\Phi'(\boldsymbol{\sigma})}{\partial x_{1n}}-\beta\frac{\partial\mathbf{G}(\mathbf{x}_1)}{\partial x_{1n}}]\mathbf{x}_2\end{bmatrix}\\
&\qquad+\frac{\partial^2\psi}{\partial\mathbf{x}_1\partial\mathbf{x}_2}\mathbf{x}_2-\frac{\partial\psi}{\partial\mathbf{x}_1}\\
&\qquad+\alpha\frac{\partial\left[\sum_{i=1}^m\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)\right]}{\partial\mathbf{x}_1}\\
&\qquad=\mathbf{0}
\end{aligned} \quad \cdots\cdots\cdots (35)$$

where $[\mathbf{Z}^1, \mathbf{Z}^2]$ and $[\mathbf{Z}^2, F]$ have been defined as follows:

$$\begin{aligned}
&[\mathbf{Z}^1,\mathbf{Z}^2]\equiv\begin{bmatrix}[Z_{n+1},Z_1]\cdots[Z_{n+1},Z_n]\\ \vdots \qquad \vdots\\ [Z_{2n},Z_1]\cdots[Z_{2n},Z_n]\end{bmatrix}\\
&[\mathbf{Z}^2,F]\equiv\begin{bmatrix}[Z_{n+1},F]\\ \vdots\\ [Z_{2n},F]\end{bmatrix}
\end{aligned} \quad \cdots\cdots (36)$$

The condition $[\mathbf{Z}^1, F]=\mathbf{0}$ becomes identical with $\mathbf{Z}^2=\mathbf{0}$.

The unknown functions $\frac{\partial\psi}{\partial\mathbf{x}_2}$ and ψ are determined from (36). As the results, we have

$$\begin{aligned}
\frac{\partial\psi}{\partial\mathbf{x}_2}&=2\{\beta\mathbf{G}(\mathbf{x}_1)-\alpha\mathbf{\Phi}'(\boldsymbol{\sigma})\}\mathbf{x}_2+\mathbf{B}\Omega(\mathbf{x}_1)\\
\psi&=\mathbf{x}_2^T\{\beta\mathbf{G}(\mathbf{x}_1)-\alpha\mathbf{\Phi}'(\boldsymbol{\sigma})\}\mathbf{x}_2\\
&\quad+\Omega^T(\mathbf{x}_1)\mathbf{B}^T\mathbf{x}_2\\
&\quad+\alpha\sum_{i=1}^m\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)\\
&=\dot{\boldsymbol{\sigma}}^T\{\text{diag}[\beta g_i(\sigma_i)-\alpha\phi_i'(\sigma_i)]\}\dot{\boldsymbol{\sigma}}\\
&\quad+\Omega^T(\mathbf{x}_1)\dot{\boldsymbol{\sigma}}+\sum_{i=1}^m\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)
\end{aligned} \quad \cdots\cdots\cdots (37)$$

where $\Omega(\mathbf{x}_1)$ is the arbitrary vector function. In (37), inequalities

$$\begin{aligned}
&\beta g_i(\sigma_i)-\alpha\phi_i'(\sigma_i)>0,\\
&\alpha\phi_i(\sigma_i)f_i(\sigma_i)>0
\end{aligned} \quad \cdots\cdots\cdots (38)$$

must be satisfied. Then we choose $\Omega(\mathbf{x}_1)$ as

$$\Omega(\mathbf{x}_1)=\begin{bmatrix}-2\sqrt{K_1(\sigma_1)\alpha\lambda_1\phi_1(\sigma_1)f_1(\sigma_1)}\\ -2\sqrt{K_2(\sigma_2)\alpha\lambda_2\phi_2(\sigma_2)f_2(\sigma_2)}\\ \vdots\\ -2\sqrt{K_m(\sigma_m)\alpha\lambda_m\phi_m(\sigma_m)f_m(\sigma_m)}\end{bmatrix} \quad \cdots (39)$$

in order for ψ to be the sum of perfect square forms, such that

$$\psi=\sum_{i=1}^m\left[\sqrt{K_i(\sigma_i)}\dot{\sigma}_i-\sqrt{\alpha\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)}\right]^2 \quad (40)$$

where, $K_i(\sigma_i)$ $(i=1,\ldots,m)$ are given as $K_i(\sigma_i)=\beta g_i(\sigma_i)-\alpha\phi_i(\sigma_i)$.

Solving (34) for $\mathbf{P}_1$ and $\mathbf{P}_2$, we obtain

$$\begin{aligned}
\mathbf{P}_1&=\alpha\mathbf{G}(\mathbf{x}_1)\mathbf{D}\{\text{diag}[\phi_i(\sigma_i)]\}\mathbf{1}\\
&\quad+\alpha\mathbf{\Phi}'(\boldsymbol{\sigma})\mathbf{x}_2+\beta\mathbf{B}\mathbf{f}(\boldsymbol{\sigma})-\mathbf{B}\Omega(\mathbf{x}_1)\\
\mathbf{P}_2&=\alpha\mathbf{D}\{\text{ diag}[\phi_i(\sigma_i)]\}\mathbf{1}+\beta\mathbf{x}_2
\end{aligned} \quad \cdots\cdots\cdots (41)$$

Choosing $\alpha=1$ and $\beta=1$, the scalar function V in (10) is given as

$$V = \frac{1}{2}\{\mathbf{x}_2 + \mathbf{D}\{\,\mathrm{diag}[\phi_i(\sigma_i)]\}\,\mathbf{1}\}^T * \{\mathbf{x}_2 + \mathbf{D}\{\,\mathrm{diag}[\phi_i(\sigma_i)]\}\,\mathbf{1}\} + \sum_{i=1}^{m}\int_0^{\sigma_i} K_i(\sigma_i)\lambda_i\phi_i(\sigma_i)d\sigma_i + \int_0^{\boldsymbol{\sigma}} \mathbf{f}^T(\boldsymbol{\sigma})d\boldsymbol{\sigma} + 2\sum_{i=1}^{m}\int_0^{\sigma_i}\sqrt{K_i(\sigma_i)\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)}d\sigma_i \quad \cdots (42)$$

The time derivative of V is of the form

$$\dot{V} = -\sum_{i=1}^{m}\left[\sqrt{K_i(\sigma_i)}\dot{\sigma}_i - \sqrt{\lambda_i\phi_i(\sigma_i)f_i(\sigma_i)}\right]^2 \quad (43)$$

Now, the conditions given in (38) result in

$$\sigma_i\phi_i(\sigma_i) > 0 \qquad (\sigma_i \neq 0)$$
$$K_i(\sigma_i) = g_i(\sigma_i) - \frac{d\phi_i(\sigma_i)}{d\sigma_i} \geq 0 \quad \cdots\cdots\cdots\cdots (44)$$

We can easily see that $\dot{V}$ in (43) satisfies Lyapunov's criteria (*iv*) and (*v*).

Next we inspect the definiteness of V in a region around the origin. According to the method in (25), we rewrite the right-hand side of (42) except the first term, such that

$$V_0 = \sum_{i=1}^{m}\int_0^{\sigma_i} H_i(\sigma_i)d\sigma_i \quad \cdots\cdots\cdots\cdots (45)$$

where

$$H_i(\sigma_i) = \lambda_i K_i(\sigma_i)\phi_i(\sigma_i) + f_i(\sigma_i) + \sqrt{\lambda_i K_i(\sigma_i)\phi_i(\sigma_i)f_i(\sigma_i)} \quad \cdots (46)$$

As $H_i(i = 1, ..., m)$ in (46) are arranged in the forms

$$H_i(\sigma_i) = \begin{cases} [\sqrt{\lambda_i K_i(\sigma_i)\phi_i(\sigma_i)} + \sqrt{f_i(\sigma_i)}]^2, & \sigma_i \geq 0 \\ -[\sqrt{-\lambda_i K_i(\sigma_i)\phi_i(\sigma_i)} - \sqrt{-f_i(\sigma_i)}]^2, & \sigma_i < 0 \end{cases} \quad \cdots\cdots (47)$$

we have

$$\sigma_i H_i(\sigma_i) \geq 0 \qquad (\sigma_i > 0) \quad \cdots\cdots\cdots\cdots (48)$$

Hence, V_0 is verified to be a positive function. Thus, the scalar function V in (42) satisfies Lyapunov's criteria (*i*)–(*iii*) as well, and is therefore a Lyapunov function of the system (1).

If we choose

$$\phi_i(\sigma_i) = \alpha'\int_0^{\sigma_i} g_i(\sigma_i)d\sigma_i, \qquad (0 \leq \alpha' \leq 1) \quad (49)$$

(42) is equivalent to that obtained in the literature [10]. The Lyapunov function given in (42) is regarded as a generalized Lyapunov function for the system (1) satisfying $\mathbf{D} = \mathbf{B}$ and $\mathbf{D}^T\mathbf{B} = \mathrm{diag}[\lambda_i]$, ($\lambda_i$: positive constants).

5.2 **Example** Let us consider a Liénard-type nonlinear system [10].

$$\ddot{y}_1 + \{g_1(\sigma_1) + g_2(\sigma_2)\}\dot{y}_1 + \{g_1(\sigma_1) - g_2(\sigma_2)\}\dot{y}_2 + f_1(\sigma_1) - f_2(\sigma_2) = 0$$
$$\ddot{y}_2 + \{g_1(\sigma_1) - g_2(\sigma_2)\}\dot{y}_1 + \{g_1(\sigma_1) + g_2(\sigma_2)\}\dot{y}_2 + f_1(\sigma_1) + f_2(\sigma_2) = 0 \quad \cdots\cdots (50)$$

where $\sigma_1 = y_1 + y_2, \quad \sigma_2 = y_2 - y_1$

We can rewrite the equation (50) in the form of (2) as

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} g_1(\sigma_1) & 0 \\ 0 & g_2(\sigma_2) \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} f_1(\sigma_1) \\ f_2(\sigma_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (51)$$

where

$$\mathbf{D} = \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{and}$$
$$\mathbf{D}^T\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \cdots\cdots\cdots\cdots (52)$$

For the system (51), the generalized Lyapunov function (42) becomes

$$V = \frac{1}{2}[x_{21} + \phi_1(\sigma_1) - \phi_2(\sigma_2)]^2 + \frac{1}{2}[x_{22} + \phi_1(\sigma_1) + \phi_2(\sigma_2)]^2 + 2\int_0^{\sigma_1} K_1(\sigma_1)\phi_1(\sigma_1)d\sigma_1 + 2\int_0^{\sigma_2} K_2(\sigma_2)\phi_2(\sigma_2)d\sigma_2 + \int_0^{\sigma_1} f_1(\sigma_1)d\sigma_1 + \int_0^{\sigma_2} f_2(\sigma_2)d\sigma_2 + 2\int_0^{\sigma_1}\sqrt{2K_1(\sigma_1)\phi_1(\sigma_1)f_1(\sigma_1)}d\sigma_1 + 2\int_0^{\sigma_2}\sqrt{2K_2(\sigma_2)\phi_2(\sigma_2)f_2(\sigma_2)}d\sigma_2 \quad \cdots\cdots\cdots\cdots (53)$$

with

$$\dot{V} = -[\sqrt{K_1(\sigma_1)}\dot{\sigma_1} - \sqrt{2\phi_1(\sigma_1)f_1(\sigma_1)}]^2$$
$$-[\sqrt{K_2(\sigma_2)}\dot{\sigma_2} - \sqrt{2\phi_2(\sigma_2)f_2(\sigma_2)}]^2 \cdots (54)$$

We can see that the Lyapunov function obtained is an extension of (22).

6. Conclusions

This paper has given the generalized Lyapunov function for the Liénard-type nonlinear system which is important as the general system expressing LRC electric circuits and spring-mass systems etc.. The Lagrange-Charpit method which is a well known technique for solving partial differential equations was applied to construct the Lyapunov function. The proposed function includes arbitrary functions, by which the Luré- type Lyapunov function is extended. The result yields all the conventional Lyapunov functions in the literature, as special cases, selecting the arbitrary functions appropriately.

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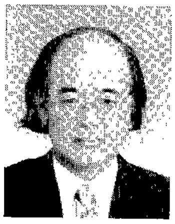

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