Paper

Evaluation of Iterative-Solutions used for Three-Dimensional Volume Integral Equation

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The impedance matrix generated by the discretization of volume integral equation is usually nonsymmetrical and dense. When the direct matrix solver such as Gaussian elimination is employed to solve the impedance matrix, $O(N^2)$ memory and $O(N^3)$ operations are required, where N is the number of unknowns. Therefore, the direct matrix solver is not suitable for the practical solution of large-scale problems by the volume integral equation. The implement of the iterative-methods is the realistic solution to improve the calculation efficiency in solving the problems. In this paper, we evaluate six well-known iterative-methods in solving the matrix equation obtained by the discretization of volume integral equation. We investigate the convergence characteristics of the residual-norms in this evaluation. In the methods based on Lanczos process, it is found that the convergence characteristics of the residual-norms become unstable under some conditions. In the methods based on Arnoldi process, it is found that the convergence characteristics of the residual-norms are always stable under various conditions. Since we found that GMRES is the most effective iterative-method in solving the matrix equation obtained by the discretization of volume integral equation, we particularly investigate the convergence characteristic of GMRES based on Arnoldi process.

Keywords:
CAD/CAE (Computer Aided Design/Computer Aided Engineering), Computational Electro-magnetics, Method of Moment, Volume Integral Equation

1. Introduction

A number of iterative-methods for the reduction of the computational costs, which can be applied to the solution in the moment-method, have been proposed so far (1) (2) (3) (4). However, the evaluation of the iterative-method used for the volume integral equation has not been reported to our knowledge. In this paper, we investigate six well-known iterative-methods in solving the matrix equation obtained by the discretization of volume integral equation. The research of the appropriate iterative-method for solving the volume integral equation, and the evaluation of the convergence characteristics under the various conditions are important problems for the numerical simulation of the large-scale scattering problems by Personal Computer (PC). We evaluate the convergence characteristics of the residual-norms of the following iterative-methods. The iterative-methods used in this evaluation are Generalized Minimal RESidual method (GMRES), Generalized Conjugate Residual method (GCR) and Orthomin (5) method that are based on Arnoldi process, and Conjugate Gradient Squared method (CGS), Bi-Conjugate Gradient STABilized method (Bi-CGSTAB) and Generalized Product-type method based on Bi-CG (GPBi-CG) that are based on Lanczos process. All of these iterative-methods have the capability to solve the nonsymmetrical and dense matrix (6).

By solving the scattering of electromagnetic wave by the dielectric rectangular plate, we investigate the most effective iterative-method to solve the volume integral equation. For the investigation of the most effective iterative-method (GMRES) that can be found by the above evaluation, we evaluate the convergence characteristics by changing the relative permittivity, the size of rectangular plate and the size of cells used in discretization that relate to the number of unknowns. The preconditioning method is not used in the evaluations in this paper. The validity of numerical values obtained and used in the evaluations has been confirmed by the optical theorem.

2. Volume Integral Equation

The outline of the volume integral equation used in this paper and the system of linear equation obtained by applying the moment-method to the volume integral equation are shown as follows (7) (8):

Electric-Field Integral Equation (EFIE) is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}^i(\boldsymbol{r}) - j\omega\mu_0 \int_V \bar{\mathbf{G}}_e(\boldsymbol{r},\boldsymbol{r}')\cdot[\epsilon_r(\boldsymbol{r}')-1]\boldsymbol{E}(\boldsymbol{r}')dv', \quad \cdots (1)$$

and we can obtain the three-dimensional volume integral equation expressed by the integration of principal value by performing the analytical integration in the finite-small region that contains singular point as

$$\boldsymbol{E}^i(\boldsymbol{r}) = -k_0^2 \int_V \bar{\mathbf{G}}_e(\boldsymbol{r},\boldsymbol{r}')\cdot[\epsilon_r(\boldsymbol{r}')-1]\boldsymbol{E}(\boldsymbol{r}')dv' + D_e(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}), \quad \cdots\cdots (2)$$

where $\boldsymbol{E}(\boldsymbol{r})$ is the total electric-field, $\boldsymbol{E}^i(\boldsymbol{r})$ is the electric-field of incident wave and $\int_V$ is the integration of principal value. Coefficient of k_0 and ϵ_r show the wave number of free space and the relative permittivity respectively, and $D_e(\boldsymbol{r})$ is given by

$$D_e(\boldsymbol{r}) = \frac{\epsilon_r(\boldsymbol{r})+2}{3}. \quad \cdots\cdots (3)$$

The electric dyadic Green's function $\bar{\mathbf{G}}_e(\boldsymbol{r},\boldsymbol{r}')$ in Eqs.(1) and (2) is given by

$$\bar{\mathbf{G}}_e(\boldsymbol{r},\boldsymbol{r}') = (\bar{\mathbf{I}} + \frac{1}{k_0^2}\nabla\nabla)g(\boldsymbol{r},\boldsymbol{r}'), \quad \cdots\cdots (4)$$

and

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{-jk_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \quad \cdots\cdots (5)$$

$$\bar{\mathbf{I}} = \hat{\boldsymbol{i}}_x\hat{\boldsymbol{i}}_x + \hat{\boldsymbol{i}}_y\hat{\boldsymbol{i}}_y + \hat{\boldsymbol{i}}_z\hat{\boldsymbol{i}}_z. \quad \cdots\cdots (6)$$

The volume of dielectric region V is divided into small cubic cells as shown in Fig. 1, L is the number of small cubic cells. We discretize Eq.(2) by using pulse function as the basis function and delta function as the weighting function (collocation method). In the discretization by the pulse function and delta function, it is well-known that accuracy of numerical results decrease in some problems [9]. However, we used this discretization method in order to solve the large-scale scattering problems, because we can drastically reduce the computational memory of our PC. The validity of the numerical calculation has been confirmed by the optical theorem.

As a result of this discretization, Eq.(2) has been approximated by a system of linear equations, which has 3L unknowns as

$$E_n^i(\boldsymbol{r}_p) = \sum_{l=1}^{L}\sum_{k=1}^{3} E_l^k Z_{pk}^{ln}$$
$$(n = 1,2,3; p = 1,2,\sim L), \quad \cdots\cdots (7)$$

where E_l^k is unknown coefficient, Z_{pk}^{ln} is impedance element, p is observation point, and n, k denote the factor of x, y, z. The matrix equation (7) can be solved by an appropriate iterative-method.

3. Iterative Solution

The significant features of six well-known iterative-methods evaluated in this paper are shown as follows.

There are some theoretical similarities between GMRES and GCR, since both iterative-methods are based on the same minimal residual condition. So they show the convergence characteristics that look alike well. Orthomin is an improved method as to the method of keeping the orthogonal vector based on GCR. Orthomin breaks and discards the orthogonal vector per k time of iterations, so it has the advantage in saving memory compared with GCR. However, the orthogonal system is broken by this manipulation, therefore the convergence of the residual-norms is not guaranteed. In our evaluation, 10 were used for the restart coefficient k.

The methods of CGS, Bi-CGSTAB and GPBi-CG are distinguished by the coefficient used in the calculation of appropriate solved vector. The method of CGS has irregular convergence characteristic. The methods of Bi-CGSTAB and GPBi-CG are improved methods for the purpose of stable convergence.

In our numerical evaluation, the convergence characteristics of these six well-known iterative-methods are discussed.

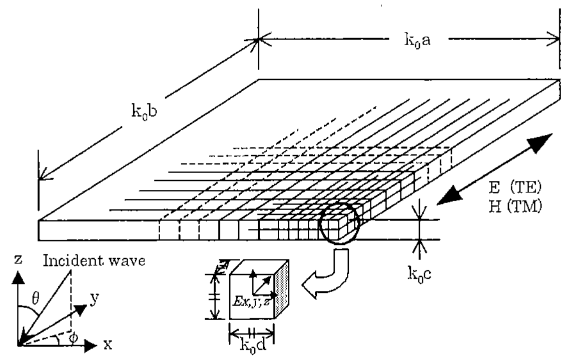

Fig. 1. The geometry of this problem. The dielectric rectangular plate is divided into small cubic cells.

Table 1. The variable range of parameters in this problem.

	$k_0a = k_0b$	k_0c	k_0d	ϵ_r	N
ex.1	6.0π	0.2π	0.2π	$2.0, 4.0, 6.0, 9.0$	2700
ex.2	6.0π	0.2π	$0.2\pi/2$	$2.0, 4.0, 6.0$	21600
ex.3	6.0π	0.2π	$0.2\pi/3$	2.0	72900
ex.4	12.0π	0.2π	0.2π	2.0	10800
ex.5	18.0π	0.2π	0.2π	2.0	24300
ex.6	24.0π	0.2π	0.2π	2.0	43200

4. Numerical Evaluation

4.1 Dielectric Rectangular Plate The geometry of this problem is shown in Fig. 1. The dielectric rectangular plate is divided into small cubic cells as shown in Fig. 1. The electric vector in each small cubic cell is assumed to be constant in our cases. Table 1 summarizes the parameters of rectangular plates. Parameters of k_0a and k_0b show the size of rectangular plates, we assume that k_0a=k_0b in this paper. Parameters of k_0c shows the thickness, k_0d shows the size of small cubic cells, and ϵ_r show the relative permittivity of rectangular plates.

The angle of incident waves in Fig. 1 is changed as $\theta = 0, 10, 20, \ldots, 90$ deg and $\phi = 0$ deg, and it is calculated on two conditions of polarization i.e., TE and TM

Table 2. The average of errors depend on optical theorem. (ex.1, $\epsilon_r = 2.0$)

Iterative-method	Average of errors(%) $\sum_{\theta=0,10,20,\ldots}^{90} error/10$
GMRES	0.049740743469
GCR	0.049740743469
Orthomin	0.049740743460
CGS	0.049740743469
Bi-CGSTAB	0.049740743468
GPBi-CG	0.049740743468

waves. The condition of incident TE wave means that the incident electric vector is parallel to the x-y plane and that of incident TM wave means that the incident magnetic vector is parallel to the x-y plane in Fig. 1.

In the problem in this paper, the impedance matrix is symmetrical and dense, since all cubic cells have the same volume.

4.2 Confirmation of Optical Theorem The validity of numerical values is confirmed by using the optical theorem. The optical theorem of this problem is shown in Fig. 1, and it can be written as follows [10];

$$\int_{S_0} |\boldsymbol{F}(\boldsymbol{r})|^2 \frac{ds}{r^2} = -4\pi Im[\boldsymbol{e}_0^* \cdot \boldsymbol{F}(\boldsymbol{r}_0)], \quad \cdots\cdots\cdots (8)$$

where $\boldsymbol{F}(\boldsymbol{r})$ represents the scattering coefficient and it is given by

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{jk^2\zeta}{4\pi}\hat{\boldsymbol{i}}_r \times \hat{\boldsymbol{i}}_r \times \int_V \boldsymbol{J}_{eq}(\boldsymbol{r}')e^{jk\boldsymbol{r}'\cdot\hat{\boldsymbol{i}}_r}dv', \quad (9)$$

and $\boldsymbol{J}_{eq}(\boldsymbol{r}')$ is given by

$$\boldsymbol{J}_{eq}(\boldsymbol{r}') = j\omega\epsilon_0[\epsilon_r(\boldsymbol{r}') - 1]\boldsymbol{E}(\boldsymbol{r}'), \quad \cdots\cdots\cdots\cdots (10)$$

where ϵ_0 is the permittivity of free space. In Eqs.(8) and (9), $\zeta = (\mu_0/\epsilon_0)^{1/2}$, S_0 is a spherical surface enclosed scattering object, $\boldsymbol{e}_0^*$ is the complex conjugate vector of incident electric-field, $\boldsymbol{r}_0$ is the direction vector of the incident waves and $\hat{\boldsymbol{i}}_r$ denotes a unit radial vector of the observation vector $\boldsymbol{r}$.

The average of errors of optical theorem obtained by various iterative methods is shown in Table 2 for the case of ex.1 in Table 1 ($\epsilon_r = 2.0$). The average of errors in Table 2 is defined as follows. We subtract the value of the left hand side of Eq.(8) from that of the right hand side of Eq.(8), divided the resultant subtraction by the right hand side of Eq.(8) and we average the results through 10 incident angles of θ.

From Table 2, we can confirm the validity of the code used in this paper, since the results are almost the same and the values are significantly small as shown in Table 2.

4.3 Evaluation of Iterative-Solutions The dependence of the convergence characteristics of each iterative-method on the relative permittivities is investigated. The dependence on the relative permittivities (ϵ_r) is shown in Figs. 2(a),(b),~(d) for the case of ex.1 in Table 1. The abscissa represents the number of iterative operations and ordinate represents the residual-norms. In Figs. 2(a),(b),~(d), each residual-norm represents the worst case among them of θ = 0, 10, 20, ..., 90 deg and ϕ = 0 deg in each iterative-method.

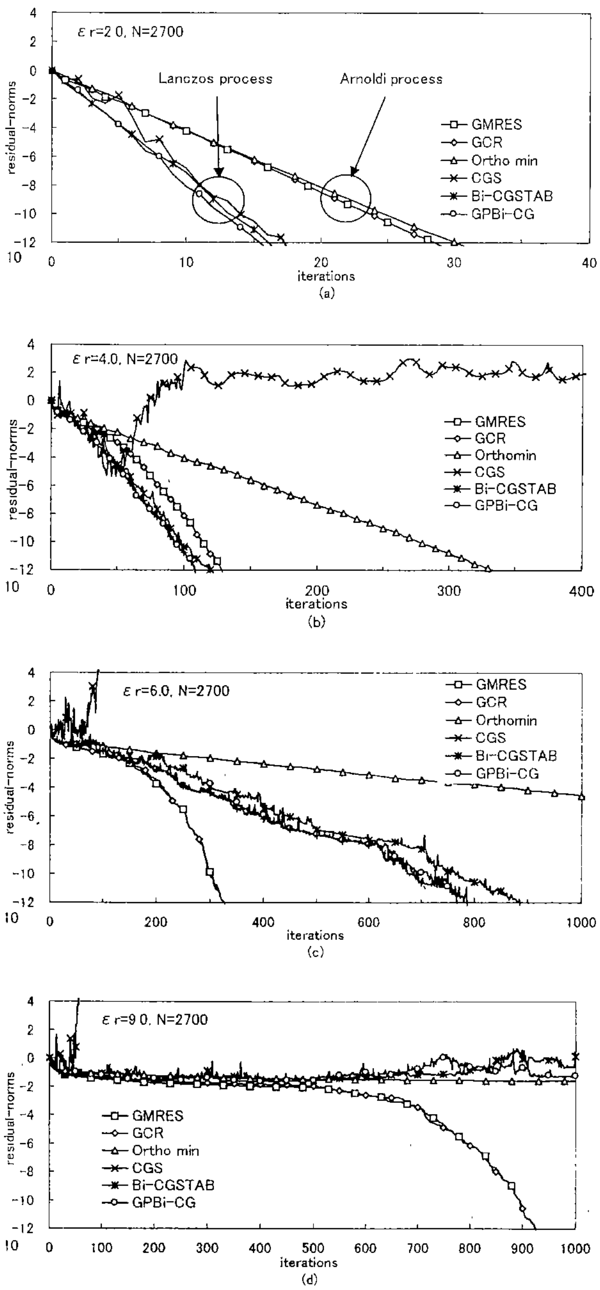

Fig. 2. The dependence on the relative permittivities of each iterative-method. (ex.1 in Table 1) (a) $\epsilon_r = 2.0$, (b) $\epsilon_r = 4.0$, (c) $\epsilon_r = 6.0$, (d) $\epsilon_r = 9.0$

Fig. 2(a) shows the characteristics in the case of small number of unknowns and rather small relative permittivity ($\epsilon_r = 2.0, N = 2700$) as ex.1 in Table 1. In Fig. 2(a), it is found that the methods based on Lanczos process i.e., CGS, Bi-CGSTAB, GPBi-CG converge about two times faster than those based on Arnoldi process i.e., GMRES, GCR, Orthomin. The convergence characteristic of CGS is irregular, and the method Bi-CGSTAB and GPBi-CG are improved to be stable compared with CGS in the convergence process.

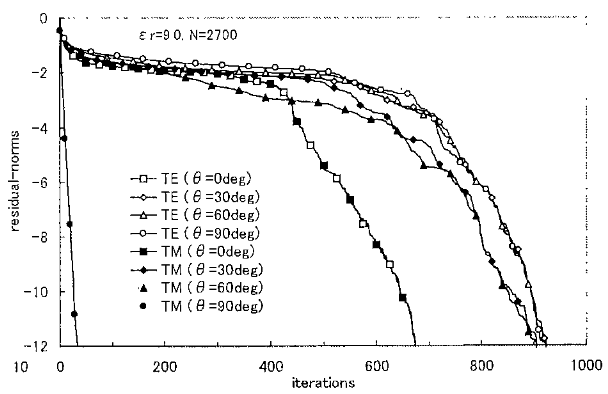

Fig. 3. The dependence on the incident angles of θ and the polarizations of incident waves by GMRES. (ex.1 in Table 1, $\epsilon_r = 9.0$)

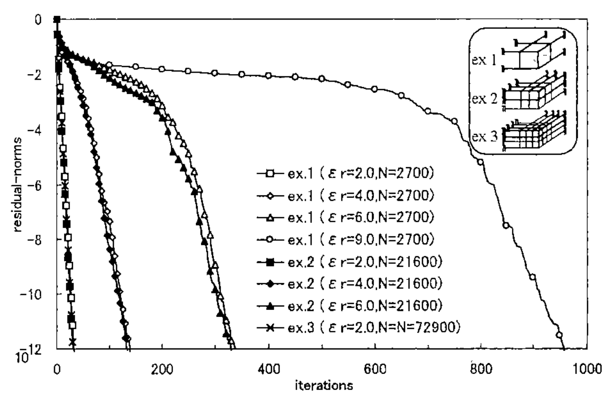

Fig. 4. The dependence on the numbers of unknowns and the relative permittivities by GMRES.

The methods based on Arnoldi process i.e., GMRES and GCR, show similar characteristics in the convergence process. These characteristics agree well to the theory. It is found that under the condition of small number of unknowns and small relative permittivity, the convergence characteristics of the methods based on Lanczos process are superior to those based on Arnoldi process in our numerical results.

Under the same condition as that used in Fig. 2(a), the convergence characteristics are investigated by increasing the relative permittivity from 4.0 to 9.0 in Figs. 2(b),(c) and (d).

From the results in Figs. 2(b),(c) and (d), it is found that the convergence characteristics become worse in the case of increasing the relative permittivity. Under the condition that the relative permittivity becomes larger than 4.0, the residual-norms of CGS are not converged. In addition, under the condition that the relative permittivity becomes 9.0 and the number of unknowns becomes 2700, it can be seen that the residual-norms of all methods except GMRES and GCR are not converged within 1000 iterative operations. These results show the superiority of GMRES and GCR based on Arnoldi process to solve the volume integral equation. Furthermore, the convergence of residual-norms of GMRES is guaranteed theoretically in any condition [11]. So, we conclude that GMRES is the most effective iterative-method for the volume integral equation, and we concentrate our investigation on the convergence characteristic of GMRES hereafter.

4.4 Evaluation of GMRES In the evaluation of GMRES, we investigate the dependence of convergence characteristic on the relative permittivity, the size of rectangular plate, and the conditions of incident wave i.e., polarization and incident angles.

Fig. 3 show the convergence characteristic under the condition of ex.1 in Table 1, and the relative permittivity is given by 9.0. The incident angle of θ is changed as $0, 30, 60, 90$ deg.

From Fig.3, it is found that the residual-norms of TE polarization are converged generally faster than TM polarization. In the case of $\theta = 90$, the residual-norms of TM polarization shows rapid convergence.

Fig. 4 shows the dependence of the convergence characteristic by GMRES on the relative permittivity (ϵ_r) and the size of cubic cells ($k_0 d$).

The sizes of the plates are the same as those of ex.1 in Table 1 ($k_0 a = k_0 b = 6.0\pi, k_0 c = 0.2\pi$). The conditions of ex.2 and ex.3 are improved conditions in the approximation of electric-field. Under the condition of ex.2, the size of small cubic cells obtained by the discretization of dielectric region is a half of that used in ex.1 ($k_0 d = 0.2\pi/2$), and under the condition of ex.3, it is one third of that used in ex.1 ($k_0 d = 0.2\pi/3$). Each residual-norm represents the worst case among them of $\theta = 0, 10, 20, \ldots, 90$ deg ($\phi = 0$ deg). When the size of rectangular plate is maintained to be constant, it is found that the number of iterative operations is mainly affected by the relative permittivity, and it is not affected by the number of unknowns from Fig. 4. Therefore, under the condition of the small relative permittivity, it is expected that we can solve the large-scale problems that have large number of unknowns by reasonable number of iterative operations when GMRES is used.

Fig. 5 shows the dependence of the convergence characteristic of GMRES on the size of rectangular plate ($k_0 a, k_0 b$) under the conditions of ex.1, ex.4, ex.5 and ex.6 in Table 1. The size of cubic cells and the relative permittivity are maintained to be same ($k_0 d = 0.2\pi, \epsilon_r = 2.0$), therefore the number of unknowns (N) is proportional to the volume of rectangular plate.

From Fig.5, it is found that the larger size of rectangular plate requires larger number of iterative operations. However, the increase of the number of iterative operations required is not proportional to the volume of rectangular plate i.e., the number of unknowns. For example, we can expect that the number of iterative operations increase only 2 times even if the volume of rectangular plate is enlarged 16 times from Fig. 5.

Fig. 6 shows the computational time and the computational memory depend on the number of unknowns. The computational memory represents the size of al-

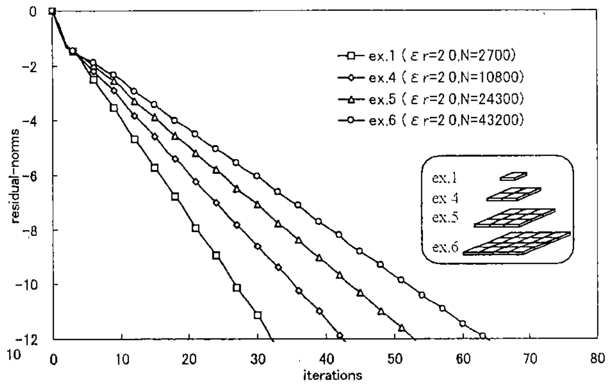

Fig. 5. The dependence on the size of rectangular plates.

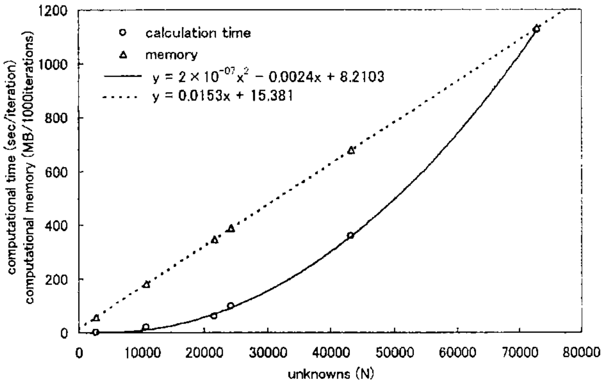

Fig. 6. The computational time and the computational memory by GMRES.

location memory used per 1000 iterative operations of GMRES. Since each component of impedance matrix can be generated in each iterative process one by one in our codes which use pulse function and point matching (collocation) method, the memory for the impedance matrix which requires $O(N^2)$ memory is not contained in this calculation.

In Fig. 6, we used a Personal Computer (PC) which have the Alpha21264 processor of 667MHz and 2GB RAM on the board. The solid line and the dotted line show the approximate functions of the computational time and the computational memory. It is found that both measured results obey to the approximate functions well as shown in Fig. 6. These characteristics are reasonable by considering theoretical basis of GMRES which requires $O(N)$ memory and $O(N^2)$ operations. From these results, it can easily anticipate that the maximum number of unknowns is about 130,000 and that the computational time is about 3,000 seconds (50 minutes) per iterative operation when full-size memory 2GB RAM is used in this system. When we calculate the large-scale problems by the volume integral equation with GMRES such as scattering of large-sized complicated composite materials, we can understand that the requirements for the computational environment are rather severe from above discussions. So, in order to perform the calculation of the large-scale problems, we must consider the methods of saving the computational time and the computational memory in addition to the investigation of improvement on the convergence characteristic.

5. Conclusions

For the numerical calculation of the scattering problem by the volume integral equation, we investigated the convergence characteristics of various iterative-methods numerically. We adopted the pulse function and the point matching (collocation) method in the discretization of the problem. The iterative-methods of GMRES, GCR and Orthomin based on Arnoldi process, and CGS, Bi-CGSTAB and GPBi-CG based on Lanczos process were evaluated concretely. It was found that GMRES is the most effective iterative-method in solving the problem by the volume integral equation under reasonable and practical situations.

The evaluation of GMRES in detail was performed. Maintaining the size of rectangular plates or the relative permittivities of rectangular plate to be same, we have found that the convergence characteristic of the residual-norms of GMRES is mainly affected by the relative permittivity and it is not affected by the number of unknowns i.e., the larger relative permittivity requires larger number of iterative operations. From the evaluation of GMRES, the maximum number of unknowns that can be treated by the conventional PC with on-board memory is not sufficient in order to solve the practical large-scale scattering problems when GMRES is adopted as the most effective iterative-method for solving the volume integral equation. So, we will investigate how to solve the volume integral equation that improves the computational time and the computational memory hereafter.

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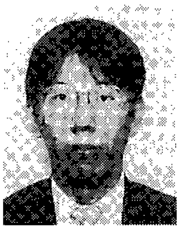

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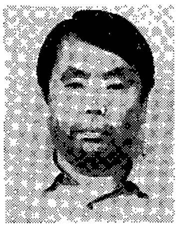

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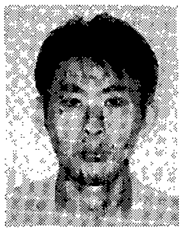

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