Paper

Robust L_2 Disturbance Attenuation for Nonlinear Systems with Input Dynamical Uncertainty

Non-member Xiaohong Jiao (Department of Mechanical Engineering, Sophia University, Japan)
Member Tielong Shen (Department of Mechanical Engineering, Sophia University, Japan)
Non-member Katsutoshi Tamura (Department of Mechanical Engineering, Sophia University, Japan)

This paper deals with the problem of robust L_2 disturbance attenuation for nonlinear systems with input dynamical uncertainty. The input dynamical uncertainty is restricted to be minimum-phase and relative degree zero. A sufficient condition is given such that the nonlinear system satisfies the L_2 gain performance and input-to-state stable (ISS) property. Using this condition, a design approach is given for smooth state feedback control law that solves the robust L_2 disturbance attenuation problem, and the approach is extended to more general case where the nominal system has higher relative degree. Finally, a numerical example is given to demonstrate the proposed approach.

Keywords: Input dynamical uncertainty, L_2 disturbance attenuation, Robust control, Input-to-state stability.

1. Introduction

In the last decade, there has been renewal of interest in developing systematic design methodologies for control of nonlinear systems. For the systems forced by disturbance, the attention was focused on the L_2 disturbance attenuation problem. In the early stage, a solution to this problem is given based on positive definite solution of Hamilton-Jacobi Inequality (HJI) [1][2][5]. Recently, it has been shown by [6][11] that if the penalty signal is of particular form, the L_2 disturbance attenuation problem can be solved by directly constructing a storage function.

For uncertain nonlinear systems, robust L_2 feedback controller based on the extended HJI was proposed by [3][13], and the constructive design method has been extended to parametric uncertain system [9][18] and to the systems with gain bounded uncertainty [14]~[16]. Also, the case where the penalty signal includes the control input term has been addressed in [17][18] by employing the constructive design method.

However, in the L_2 disturbance attenuation approaches, the stability was considered only for the system unforced by the disturbance. As is well-known, in nonlinear systems, the asymptotical stability of free system does not necessarily imply the boundedness of the state when the system is forced by bounded disturbance [12]. Indeed, for describing this boundedness property of a system under bounded input, the notion of ISS has been proposed by [7], and it has been shown that a necessary and sufficient condition for ISS can be given by a dissipation inequality [8][12]. Using this result, we are able to put the ISS specification to the dissipation inequality-based L_2 disturbance attenuation approach. Recently, along this research line, the L_2 disturbance attenuation with ISS property has been studied by [12].

On the other hand, in the field of robust control of nonlinear systems, the attention has been focused on a broader class of uncertainties. Robust control of nonlinear systems with input dynamical uncertainty has been investigated by many researchers (see [10][19] and the references therein). In [10], a dynamical state feedback control law is proposed to solve the robust stabilization problem under the assumption that the uncertainty is minimum-phase and relative degree zero. However, the approach requires a priori knowledge about the stability margin of the uncertainty. A static feedback control law is designed by [19]. In [19], it has been shown that the nonlinear systems with input dynamical uncertainty can be transformed into feedback loop structure, and the state feedback stabilizing control law is given based on gain assignment techniques [21], which is a successful application of the small gain theorem.

In this paper we focus our attention on robust L_2 disturbance attenuation problem for nonlinear systems with input dynamical uncertainty. The uncertainty considered in this paper is the same class as shown in [19]. However, our goal is not only robust stability but also robust L_2 gain performance and ISS property. Then, a feedback controller will be derived that solves the robust L_2 problem. Furthermore, the design method will be extended to more general system with relative degree larger than one. Finally, we will show a numerical example.

2. Preliminaries

We consider the systems with input dynamical uncertainty described by the following form

$$\left\{\begin{array}{l} \dot{x} = f(x) + g_1(x)w + g_2(x)v \\ z = h(x) \\ v = c(\xi) + u \\ \dot{\xi} = A(\xi) + bu \end{array}\right. \qquad \cdots\cdots\cdots\cdots (1)$$

where $x \in R^n$, $w \in R^r$ and $z \in R^q$ denote the state, disturbance and penalty signal, respectively. $f(x)$, $g_1(x)$, $g_2(x)$ and $h(x)$ are known smooth vector fields, $f(0)=0$, $h(0)=0$. $\xi \in R^p$ is the state of uncertain dynamics driven by the control input $u \in R$. $v \in R$ is the output of the uncertainty. $A(\cdot)$ and $c(\cdot)$ are unknown vector field and function vanishing at the origin, respectively, and b is an unknown vector bounded by $\|b\| \le b_0$, where b_0 is a known constant.

If there is no uncertain dynamics in the path from u to v, i.e. $v=u$, the system (1) is represented by

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h(x) \end{cases} \quad \cdots\cdots\cdots\cdots (2)$$

We call this system as the nominal system of (1), and we say the uncertainty is admissible, if the input dynamical uncertainty satisfies the condition described in the next section (see A1).

For the system (1), the robust L_2 disturbance attenuation problem with ISS is given as follows. For any given $\gamma > 0$, find a smooth feedback control law

$$u = \alpha(x) \quad \cdots\cdots\cdots\cdots (3)$$

with $\alpha(0)=0$ such that for all admissible input dynamical uncertainty, the following conditions are satisfied.

[P1] The closed-loop system is input-to-state stable with respect to the disturbance input w.

[P2] The closed-loop system has L_2 gain less than or equal to γ, from the disturbance input w to the penalty output z, i.e. for $x(0)=0$ and $\xi(0)=0$,

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt, \quad \forall w \quad \cdots (4)$$

holds, where $T>0$ is any given scalar.

[P3] The origin $(x,\xi)=(0,0)$ of the free system unforced by the disturbance is globally asymptotically stable.

In order to obtain our main result, first, we consider the system described by

$$\begin{cases} \dot{x} = f_c(x) + g_c(x)w \\ z = h_c(x) \end{cases} \quad \cdots\cdots\cdots\cdots (5)$$

where $x \in R^n$, $w \in R^r$, $z \in R^q$. $f_c(x)$, $g_c(x)$ and $h_c(x)$ are smooth vector fields, $f_c(0)=0$, $h_c(0)=0$.

As is well-known [7], the system (5) is said to be ISS, if there exist a $\mathcal{KL}_\infty$ function β and a $\mathcal{K}_\infty$ function χ such that

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \chi(\|w\|), \quad \forall t \ge 0 \quad \cdots\cdots (6)$$

holds for any w.

According to the necessary and sufficient condition given in [8], the system (5) is ISS if and only if there exist $\mathcal{K}_\infty$ functions λ and κ, such that the inequality

$$\frac{\partial V}{\partial x}\left[f_c(x) + g_c(x)w\right] \le \lambda(\|w\|) - \kappa(\|x\|), \quad \forall x \quad (7)$$

holds for any w.

Moreover, by the relation between L_2 gain and γ-dissipativity, the system (5) satisfies (4) for any given $T>0$, if there exists a positive definite function $V(x)$ such that the following inequality is satisfied.

$$\frac{\partial V}{\partial x}\left[f_c(x) + g_c(x)w\right] \le \frac{1}{2}\left\{\gamma^2\|w\|^2 - \|z\|^2\right\}, \quad \forall w \quad (8)$$

Comparing the dissipation inequalities (7) and (8), it is obvious that the system (5) has L_2 gain, which is less than or equal to $\gamma > 0$, with ISS, if the positive definite function $V(x)$ satisfies

$$\frac{\partial V}{\partial x}\left[f_c(x) + g_c(x)w\right] \le \frac{1}{2}\left\{\gamma^2\|w\|^2 - \|z\|^2\right\} - \kappa(\|x\|), \quad \forall x, \quad \forall w \quad (9)$$

Summarizing the argument above, we have the following Lemma 1 which is a technical Lemma for our main results.

Lemma 1. *For any given $\gamma > 0$, if there exist a positive definite function $V(x)$ ($V(0)=0$) and a $\mathcal{K}_\infty$ function κ, such that the Hamilton-Jacobi Inequality (HJI)*

$$\frac{\partial V}{\partial x} f_c(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} g_c(x) g_c^T(x) \frac{\partial^T V}{\partial x} + \frac{1}{2} h_c^T(x) h_c(x) + \kappa(\|x\|) \le 0, \quad \forall x \quad \cdots\cdots\cdots (10)$$

holds, then the system (5) satisfies [$P1$]-[$P3$].

Proof. It is easy to show that $V(x)$ satisfies HJI (10) if and only if $V(x)$ satisfies the dissipation inequality (9). Thus, [$P1$] and [$P2$] follow from (7) and (8), respectively. Finally, when $w=0$, following by (7), $V(x)$ satisfies $L_{f_c}V(x) < 0$ for all nonzero x. Therefore, the system $\dot{x} = f_c(x)$ is globally asymptotically stable at equilibrium $x=0$, i.e. [$P3$] is satisfied. ∎

3. Main result

We now consider the robust L_2 disturbance attenuation problem with ISS in Section 2.

First, we consider the systems (1) with $n=1$, i.e. the state x is a scalar variable. Suppose the system (1) satisfies the following assumptions.

[A1] There exists a known smooth $\mathcal{K}_\infty$ function $\hat{c}(\cdot)$ such that $\|c(\xi)\| \le \hat{c}(\|\xi\|)$, $\forall \xi \in R^p$. For the zero dynamics of ξ-subsystem disturbed by (d_1, d_2)

$$\dot{\xi} = A_0(\xi + d_1) + d_2 \quad \cdots\cdots\cdots\cdots (11)$$

there exist a positive definite function $W(\xi)$ and $\mathcal{K}_\infty$ functions $\beta_1(\cdot)$, $\beta_2(\cdot)$ and $\beta_3(\cdot)$ such that

$$\frac{\partial W(\xi)}{\partial \xi}\left[A_0(\xi + d_1) + d_2\right] \le -\beta_1(\|\xi\|) + \beta_2(\|d_1\|) + \beta_3(\|d_2\|), \quad \forall d_1, d_2 \quad (12)$$

holds, where $A_0(\xi) := A(\xi) - bc(\xi)$.

This assumption means that the zero dynamics of ξ-subsystem is ISS with respect to the inputs (d_1, d_2).

[A2] There exist constants $g_0 > 0$ and $\hat{g}_1 > 0$ such that the following inequalities hold

$$g_2(x) \ge g_0, \quad \|g_1(x)w\| \le \hat{g}_1\|w\|, \quad \forall x \quad \cdots (13)$$

Now, we employ the change of coordinate used in [19]

$$\bar{\xi} = \xi - b\int_0^x g_2^{-1}(s)ds \quad \cdots\cdots\cdots\cdots (14)$$

to transform the system (1) into the following form

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)(u + y_0) \\ z = h(x) \\ y_0 = c(\bar{\xi} + b\int_0^x g_2^{-1}(s)ds) \\ \dot{\bar{\xi}} = A_0(\bar{\xi} + d_1) + d_2 \end{cases} \quad \cdots (15)$$

where

$$d_1 = b\int_0^x g_2^{-1}(s)ds$$
$$d_2 = -bg_2{}^{-1}(x)\left[f(x) + g_1(x)w\right]$$

Moreover, according to the assumptions [$A1$] and [$A2$], function $c(\cdot)$ satisfies the following inequality

$$\|c(\bar{\xi} + b\textstyle\int_0^x g_2^{-1}(s)ds)\| \le \hat{c}(\|\bar{\xi} + b\textstyle\int_0^x g_2^{-1}(s)ds\|)$$
$$\le k_1(\|\bar{\xi}\|) + k_2(\|x\|), \quad \forall x \quad \cdots\cdots\cdots\cdots\cdots (16)$$

where $k_1(\cdot)$ and $k_2(\cdot)$ are $\mathcal{K}_\infty$ functions defined by

$$k_1(\|\bar{\xi}\|) = \hat{c}(2\|\bar{\xi}\|), \quad k_2(\|x\|) = \hat{c}(\frac{2}{g_0}b_0\|x\|).$$

For the system (15), we now consider the feedback control law (3). Denote $f_c(x) = f(x) + g_2(x)\alpha(x)$.

Obviously, there exists a smooth class $\mathcal{K}_\infty$ function $\hat{f}(\cdot)$ such that the following inequality holds

$$\|bf(x)g_2^{-1}(x)\| \le b_0\hat{f}(\|x\|), \quad \forall x$$

Define a $\mathcal{K}_\infty$ function $\hat{\beta}_2(\cdot)$ as follows

$$\hat{\beta}_2(\|x\|) = \beta_2(g_0^{-1}b_0\|x\|) + 2lb_0^2\hat{f}^2(\|x\|) \quad \cdots\cdot (17)$$

where l is a positive scalar. Then, we can find $\mathcal{K}_\infty$ functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfying

$$\nu(2k_1(\|\bar{\xi}\|)) \le \frac{\gamma^2}{4\rho}\beta_1(\|\bar{\xi}\|) - \varepsilon_1\|\bar{\xi}\|^2, \quad \forall\bar{\xi} \quad \cdots\cdots (18)$$

$$\kappa(\|x\|) \ge \frac{\gamma^2}{4\rho}\hat{\beta}_2(\|x\|) + \nu(2k_2(\|x\|)) + \varepsilon_2\|x\|^2, \quad \forall x \quad (19)$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are sufficient small constants, ρ is a positive constant.

Lemma 2. *Suppose that there exists a scalar $l > 0$ such that the $\mathcal{K}_\infty$ function $\beta_3(\cdot)$ satisfies*

$$\beta_3(\|d_2\|) \le l\|d_2\|^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)$$

Consider the closed-loop system (15) with (3). If for any given $\gamma > 0$, there exists a positive definite function $U(x)$ such that the dissipation inequality

$$\frac{\partial U(x)}{\partial x}\left[f_c(x) + g_1(x)w + g_2(x)y_0\right] \le$$
$$\frac{1}{2}\left\{\frac{\gamma^2}{2}\|w\|^2 - \|z\|^2\right\} + \nu(\|y_0\|) - \kappa(\|x\|), \quad \forall x \quad (21)$$

holds for any w, then the closed-loop system satisfies [P1]-[P3] for all admissible uncertainty.

Proof. By the assumption (12), we have

$$\dot{W}(\bar{\xi}) \le -\beta_1(\|\bar{\xi}\|) + \beta_2(\|d_1\|) + l\|d_2\|^2$$
$$\le -\beta_1(\|\bar{\xi}\|) + \beta_2\left(\left\|b\int_0^x g_2^{-1}(s)ds\right\|\right)$$
$$+ l\|bg_2{}^{-1}(x)\left[f(x) + g_1(x)w\right]\|^2$$
$$\le -\beta_1(\|\bar{\xi}\|) + \beta_2\left(\left\|b\int_0^x g_2^{-1}(s)ds\right\|\right)$$
$$+ 2l\|bf(x)g_2^{-1}(x)\|^2 + 2l\|bg_1(x)g_2^{-1}(x)w\|^2$$

Moreover, from the assumption (13), we have

$$\|b\textstyle\int_0^x g_2^{-1}(s)ds\| \le \frac{1}{g_0}b_0\|x\|,$$
$$\|bg_1(x)g_2^{-1}(x)w\| \le \frac{\hat{g}_1}{g_0}b_0\|w\|$$

Hence, by (17) and choosing $\rho = 2l\frac{\hat{g}_1^2}{g_0^2}b_0^2$, we get

$$\dot{W}(\bar{\xi}) \le -\beta_1(\|\bar{\xi}\|) + \hat{\beta}_2(\|x\|) + \rho\|w\|^2 \quad \cdots\cdots (22)$$

Note that the closed-loop system can be represented by

$$\dot{\tilde{x}} = F(\tilde{x}, w) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (23)$$

where

$$\tilde{x} = \begin{bmatrix} \bar{\xi} \\ x \end{bmatrix}, \; F(\tilde{x}, w) = \begin{bmatrix} A_0(\bar{\xi} + d_1) + d_2 \\ f_c(x) + g_1(x)w + g_2(x)y_0 \end{bmatrix}$$

For the given $\gamma > 0$, choose a positive definite function

$$V(\tilde{x}) = \frac{\gamma^2}{4\rho}W(\bar{\xi}) + U(x) \quad \cdots\cdots\cdots\cdots\cdots\cdots (24)$$

as a candidate of the storage function for the closed-loop system (23). Using (21) and (22), the time derivative of V along any trajectories of (23) satisfies

$$\dot{V} = \begin{bmatrix} \frac{\gamma^2}{4\rho}\frac{\partial W}{\partial\bar{\xi}} & \frac{\partial U}{\partial x} \end{bmatrix} \begin{bmatrix} A_0(\bar{\xi} + d_1) + d_2 \\ f_c(x) + g_1(x)w + g_2(x)y_0 \end{bmatrix}$$
$$\le -\frac{\gamma^2}{4\rho}\beta_1(\|\bar{\xi}\|) + \frac{\gamma^2}{4\rho}\hat{\beta}_2(\|x\|) + \frac{\gamma^2}{2}\|w\|^2$$
$$-\frac{1}{2}\|z\|^2 + \nu(\|y_0\|) - \kappa(\|x\|) \quad \cdots\cdots\cdots (25)$$

Substituting (16) into (25) obtains

$$\dot{V} \le -\frac{\gamma^2}{4\rho}\beta_1(\|\bar{\xi}\|) + \frac{\gamma^2}{4\rho}\hat{\beta}_2(\|x\|) + \frac{\gamma^2}{2}\|w\|^2$$
$$-\frac{1}{2}\|z\|^2 + \nu(k_1(\|\bar{\xi}\|) + k_2(\|x\|)) - \kappa(\|x\|)$$
$$\le \left\{-\frac{\gamma^2}{4\rho}\beta_1(\|\bar{\xi}\|) + \nu(2k_1(\|\bar{\xi}\|))\right\} + \frac{\gamma^2}{2}\|w\|^2 - \frac{1}{2}\|z\|^2$$
$$+ \left\{\frac{\gamma^2}{4\rho}\hat{\beta}_2(\|x\|) + \nu(2k_2(\|x\|)) - \kappa(\|x\|)\right\} \cdots (26)$$

It is easy to show that by choosing $\mathcal{K}_\infty$ functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfying (18) and (19), respectively, the inequality

$$\dot{V} \le \frac{1}{2}\left\{\gamma^2\|w\|^2 - \|z\|^2\right\} - \hat{\kappa}(\|\tilde{x}\|) \quad \cdots\cdots\cdots (27)$$

holds, where $\hat{\kappa}(\|\tilde{x}\|) = \varepsilon\|\tilde{x}\|^2$ and $\varepsilon \le min\{\varepsilon_1, \varepsilon_2\}$.

Therefore, by Lemma 1 we conclude that the closed-loop system satisfies [P1]-[P3] for all admissible uncertainty. ∎

Remark 1. Note that the uncertain dynamics in (15) is driven by both of the disturbance w and the state x. As is shown by (22), under the assumption [$A1$], the zero dynamics is also ISS with respect to x and w. Furthermore, if there exist constants $\mu_1 > 0$ and $\mu_2 > 0$ such that $\mu_1\|\bar{\xi}\|^2 \leq \beta_1(\|\bar{\xi}\|)$ and $\hat{\beta}_2(\|x\|) \leq \mu_2\|x\|^2$ hold, then, along any trajectories of the zero dynamics, we have

$$\dot{W} \leq -\mu_1\|\bar{\xi}\|^2 + \mu_3\left\|\begin{bmatrix} w \\ x \end{bmatrix}\right\|^2$$

where $\mu_3 \geq max\{\mu_2, \rho\}$. This means that the zero dynamics has finite L_2 gain from $[w \ \ x]^T$ to $\bar{\xi}$, which is less than $\sqrt{\mu_3/\mu_1} > 0$. This kind of uncertainty with finite L_2 gain has been addressed by many papers to solve robust stabilization problem. However, we do not require the uncertain dynamics to have finite L_2 gain, though our goal contains the asymptotic stability $[P3]$.

Using the condition given in Lemma 2, a solution of the robust L_2 disturbance attenuation problem with ISS can be found for the system (1).

Theorem 1. *Consider nonlinear system (1) satisfying assumptions* $[A1]$ *and* $[A2]$*, and suppose that there exists a positive constant* ϵ *such that*

$$\frac{\epsilon^2}{2}\|y_0\|^2 \leq \nu(\|y_0\|), \quad \forall y_0 \quad \cdots\cdots\cdots\cdots\cdots\cdots (28)$$

holds. For any given $\gamma > 0$*, if there exists a solution* $U(x) > 0$ $(U(0) = 0)$ *to HJI*

$$\frac{\partial U}{\partial x}f(x) + \frac{1}{\gamma^2}\frac{\partial U}{\partial x}g_1(x)g_1^T(x)\frac{\partial^T U}{\partial x} + \frac{1}{2}h^T(x)h(x)$$
$$-\frac{1}{2\epsilon^2}\frac{\partial U}{\partial x}g_2(x)g_2^T(x)\frac{\partial^T U}{\partial x} + \kappa(\|x\|) \leq 0 \ \cdots\cdots (29)$$

then a solution of the robust L_2 *disturbance attenuation problem with ISS is given by*

$$u = \alpha(x) = -\frac{1}{\epsilon^2}g_2^T(x)\frac{\partial^T U}{\partial x}(x) \ \cdots\cdots\cdots\cdots (30)$$

where $\nu(\cdot)$ *and* $\kappa(\cdot)$ *are any* $\mathcal{K}_\infty$ *functions satisfying (18) and (19), respectively.*

Proof. Note that the closed-loop system of (1) with the controller (30) is described by (23). We will show that under the condition (28), the positive solution $U(x)$ in (29) will satisfy the condition in Lemma 2.

Using (29), a straightforward calculation gives

$$\frac{\partial U(x)}{\partial x}\{f(x) + g_2(x)\alpha(x) + g_1(x)w + g_2(x)y_0\}$$
$$\leq -\frac{1}{\gamma^2}\frac{\partial U}{\partial x}g_1(x)g_1^T(x)\frac{\partial^T U}{\partial x} - \frac{1}{2}h^T(x)h(x)$$
$$+\frac{1}{2\epsilon^2}\frac{\partial U}{\partial x}g_2(x)g_2^T(x)\frac{\partial^T U}{\partial x} - \kappa(\|x\|)$$
$$+\frac{\partial U(x)}{\partial x}\{g_2(x)\alpha(x) + g_1(x)w + g_2(x)y_0\}$$
$$\leq \frac{\gamma^2}{4}\|w\|^2 + \frac{\epsilon^2}{2}\|y_0\|^2 + \frac{1}{\epsilon^2}\frac{\partial U}{\partial x}g_2(x)g_2^T(x)\frac{\partial^T U}{\partial x}$$
$$+\frac{\partial U}{\partial x}g_2(x)\alpha(x) - \frac{1}{2}\|z\|^2 - \kappa(\|x\|)$$

Taking (28) and (30) into account we obtain

$$\frac{\partial U(x)}{\partial x}\{f(x) + g_2(x)\alpha(x) + g_1(x)w + g_2(x)y_0\}$$
$$\leq \frac{1}{2}\left\{\frac{\gamma^2}{2}\|w\|^2 - \|z\|^2\right\} + \nu(\|y_0\|) - \kappa(\|x\|) \ (31)$$

Therefore, by Lemma 2 we conclude that the closed-loop system satisfies $[P1] - [P3]$. ∎

We now consider the robust stabilization problem for the system (1) with $w = 0$. As specified in P3, Theorem 1 presents a robust stabilizing controller. In fact, the robust stability follows by the small gain theorem, because if $U(x)$ satisfies (29) and $w = 0$, then $U(x)$, as shown in (31), will satisfy the following dissipation inequality

$$\frac{\partial U}{\partial x}\{f(x) + g_2(x)(\alpha(x) + y_0)\} \leq \nu(\|y_0\|) - \kappa(\|x\|) \ (32)$$

The inequality (32) is nothing but a gain condition for the x-subsystem of (15) from the input y_0 to the output x. Thus, the small gain condition[21] is satisfied with the gain constraint on the $\bar{\xi}$-subsystem described by $[A1]$.

Obviously, we can obtain a robust controller by directly constructing a function $U(x)$ that satisfies (32). As a special case, if we choose a candidate for the storage function $U(x)$ as

$$U(x) = \frac{1}{2}x^2 \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (33)$$

then a robust stabilizing controller can be given as follows.

Corollary 1. *Consider the system (1) with* $w = 0$*. Suppose the uncertainty satisfies* $[A1]$ *and* $[A2]$*. If there exists a positive constant* $\epsilon > 0$ *such that the inequality (28) holds, then a feedback controller which renders the closed-loop system robust asymptotically stable is given by*

$$u = \alpha(x) = -g_2^{-1}(x)(f(x) + \tilde{\kappa}(x)) - \frac{1}{2\epsilon^2}g_2^T(x)x \ (34)$$

where $\tilde{\kappa}(x)$ *is a function satisfying* $\kappa(\|x\|) = \tilde{\kappa}(x)x$.

Proof. Let $U(x)$ be given by (33). Along any trajectory of the system (1) when $w = 0$, we have

$$\frac{\partial U(x)}{\partial x}\{f(x) + g_2(x)\alpha(x) + g_2(x)y_0\}$$
$$\leq xg_2(x)[\alpha(x) + g_2^{-1}(x)f(x) + \frac{1}{2\epsilon^2}g_2^T(x)x] + \frac{\epsilon^2}{2}\|y_0\|^2 \ (35)$$

Substituting (34) into (35) and considering the inequality (28), we can obtain the inequality (32). Moreover, using the same argument as the proof of Lemma 2, the robust stability of the closed loop system is guaranteed by the Lyapunov function

$$V(x, \bar{\xi}) = \frac{\gamma^2}{4\rho}W(\bar{\xi}) + \frac{1}{2}x^2 \ \cdots\cdots\cdots\cdots\cdots\cdots (36)$$

since in this case we can easily show that

$$\dot{V} \leq -\hat{\kappa}(\|\tilde{x}\|) \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (37)$$

∎

Remark 2. It should be noted that the same Lyapunov function (33) has been employed by[19] to construct a robust stabilizing controller for system (1) when $w = 0$, and the robust stability follows also by the small gain theorem. In[19] a robust stabilizing controller is given by

$$u = -x\hat{\alpha}(x) - 2\gamma_x^{-1}(\|x\|)sgn(x) \quad \cdots\cdots\cdots (38)$$

where $\hat{\alpha}(x)$ is a smooth function satisfying the inequality

$$\| - g_2^{-1}(x)[kx + f(x)]\| \leq \|x\|\hat{\alpha}(x)$$

and $\gamma_x(\cdot)$ is a $\mathcal{K}_\infty$ function which represents a gain of the x-subsystem from the input y_0 to the output x.

Comparing the controller (38) with (34), it is easy to find that two controllers have almost same gain terms, however, (34) consists of smooth functions only.

We consider the systems (1) with $n > 1$, i.e. $x = [x_1 \ \dots \ x_n]^T$. Suppose that we can define an output $y = h_a(x)$ such that the nominal system has the relative degree one. Then under certain geometrical conditions [4], there exists a coordinate transform

$$\begin{bmatrix} \zeta \\ y \end{bmatrix} = \phi(x) = \begin{bmatrix} \phi_0(x) \\ h_a(x) \end{bmatrix}$$

such that the system is transformed to:

$$\begin{cases} \dot{\zeta} = f_1(\zeta, y) + p_1(\zeta, y)w \\ \dot{y} = f_2(\zeta, y) + p_2(\zeta, y)w + g(\zeta, y)v \\ z = h(\zeta, y) \\ v = c(\xi) + u \\ \dot{\xi} = A(\xi) + bu \end{cases} \quad \cdots\cdots (39)$$

where $\zeta \in R^{n-1}$, $y \in R$. $f_1(\zeta, y)$, $p_1(\zeta, y)$, $p_2(\zeta, y)$ and $h(\zeta, y)$ are smooth vector fields, $f_2(\zeta, y)$ and $g(\zeta, y)$ are smooth functions, $f_1(0, 0) = 0$, $f_2(0, 0) = 0$, $h(0, 0) = 0$, $g(\zeta, y) \neq 0, \forall \zeta, y$.

In the following, we address the robust L_2 disturbance attenuation problem with ISS for the system (39). Suppose that the input dynamical uncertainty satisfies assumption [$A1$] and the nominal system of (39) satisfies the following condition.

[**A2′**] There exist constants $\hat{g} > 0$ and $\hat{p} > 0$ such that the following inequalities hold

$$g(\zeta, y) \geq \hat{g}, \quad \|p_2(\zeta, y)w\| \leq \hat{p}\|w\|, \quad \forall \zeta, y \ (40)$$

The change of coordinate

$$\bar{\xi} = \xi - b\int_0^y g^{-1}(\zeta, s)ds \quad \cdots\cdots\cdots\cdots\cdots\cdots (41)$$

transforms the system (39) to the following form

$$\begin{cases} \dot{\zeta} = f_1(\zeta, y) + p_1(\zeta, y)w \\ \dot{y} = f_2(\zeta, y) + p_2(\zeta, y)w + g(\zeta, y)(u + y_0) \\ y_0 = c(\bar{\xi} + b\int_0^y g^{-1}(\zeta, s)ds) \\ z = h(\zeta, y) \\ \dot{\bar{\xi}} = A_0(\bar{\xi} + d_1) + d_2 \end{cases} \quad (42)$$

where

$$d_1 = b\int_0^y g^{-1}(\zeta, s)ds$$
$$d_2 = -bg^{-1}(\zeta, y)\left[f_2(\zeta, y) + p_2(\zeta, y)w\right].$$

Observe that the functions $f_1(\zeta, y)$, $p_1(\zeta, y)$ and $h(\zeta, y)$ can be decomposed to

$$\begin{cases} f_1(\zeta, y) = f_1(\zeta, 0) + \tilde{f}_1(\zeta, y)y \\ p_1(\zeta, y) = p_1(\zeta, 0) + \tilde{p_1}(\zeta, y)y \\ h^T(\zeta, y)h(\zeta, y) = h^T(\zeta, 0)h(\zeta, 0) + H(\zeta, y)y \end{cases} \quad (43)$$

Moreover, from assumption [$A2'$], we have

$$\left\| b\int_0^y g^{-1}(\zeta, s)ds \right\| \leq \frac{1}{\hat{g}}b_0\|y\|,$$
$$\|bg^{-1}(\zeta, y)p_2(\zeta, y)w\| \leq \frac{\hat{p}}{\hat{g}}b_0\|w\|$$

and the inequality $\|bg^{-1}(\zeta, y)f_2(\zeta, y)\| \leq b_0\hat{f}_2(\|\eta\|)$ holds for some smooth $\mathcal{K}_\infty$ function $\hat{f}_2(\cdot)$, $\eta = [\zeta \ \ y]^T$.

By choosing $\hat{\beta}_2(\|\eta\|) = \beta_2(\hat{g}^{-1}b_0\|y\|) + 2lb_0{}^2\hat{f}_2^2(\|\eta\|)$ and $\rho = 2l\hat{p}^2\hat{g}^{-2}b_0{}^2$, the inequality

$$\frac{\partial W(\bar{\xi})}{\partial \bar{\xi}}\dot{\bar{\xi}} \leq -\beta_1(\|\bar{\xi}\|) + \hat{\beta}_2(\|\eta\|) + \rho\|w\|^2$$

holds for the input uncertainty.

According to assumptions [$A1$] and [$A2'$], function $c(\cdot)$ satisfies the following inequality

$$\left\|c(\bar{\xi} + b\textstyle\int_0^y g^{-1}(\zeta, s)ds)\right\| \leq \hat{c}(\|\bar{\xi} + b\textstyle\int_0^y g^{-1}(\zeta, s)ds\|)$$
$$\leq k_1(\|\bar{\xi}\|) + k_2(\|y\|), \quad \forall y \quad \cdots\cdots\cdots\cdots (44)$$

where $k_1(\cdot)$ and $k_2(\cdot)$ are $\mathcal{K}_\infty$ functions defined by

$$k_1(\|\bar{\xi}\|) = \hat{c}(2\|\bar{\xi}\|), \quad k_2(\|y\|) = \hat{c}(\frac{2}{\hat{g}}b_0\|y\|).$$

Theorem 2. *Consider the system (39). Suppose the assumptions [A1] and [A2′] hold. For any given $\gamma > 0$, if there exist a positive definite function $U_1(\zeta)$ such that*

$$\frac{\partial U_1}{\partial \zeta}f_1(\zeta, 0) + \frac{1}{\gamma^2}\frac{\partial U_1}{\partial \zeta}p_1(\zeta, 0)p_1^T(\zeta, 0)\frac{\partial^T U_1}{\partial \zeta}$$
$$+h^T(\zeta, 0)h(\zeta, 0) + \kappa_1(\|\zeta\|) \leq 0, \quad \forall \zeta \quad \cdots\cdots (45)$$

holds, then a solution to the robust L_2 disturbance attenuation problem with ISS is given by

$$u = \alpha(\zeta, y) = g^{-1}(\zeta, y)\left\{\alpha_1(\zeta, y) - f_2(\zeta, y)\right\} \ (46)$$

where

$$\alpha_1(\zeta, y) = -\frac{\partial U_1}{\partial \zeta}\tilde{f}_1(\zeta, y) - \frac{1}{4\epsilon^2}g^2(\zeta, y)y - H(\zeta, y)$$
$$-\frac{2}{\gamma^2}y\left[\frac{\partial U_1}{\partial \zeta}\tilde{p_1}(\zeta, y)\tilde{p_1}^T(\zeta, y)\frac{\partial^T U_1}{\partial \zeta} + p_2{}^2(\zeta, y)\right]$$
$$-(\kappa_2(\|y\|) + k)y \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (47)$$

ϵ is any positive constant satisfying (28), k is a positive constant, $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are any $\mathcal{K}_\infty$ functions satisfying the following conditions

$$\kappa_1(s) \geq 2\kappa(2s), \quad (\kappa_2(s) + k)s^2 \geq 2\kappa(2s), \quad \forall s > 0 \ (48)$$

$\kappa(\cdot)$ is any $\mathcal{K}_\infty$ function satisfying (19).

Proof. Note that the closed-loop system can be represented as follows:

$$\begin{cases} \dot{\eta} = F(\eta) + P(\eta)w + G(\eta)y_0 \\ y_0 = c(\bar{\xi} + b\int_0^y g^{-1}(\zeta, s)ds) \\ z = h(\zeta, y) \\ \dot{\bar{\xi}} = A_0(\bar{\xi} + d_1) + d_2 \end{cases} \quad \cdots\cdots\cdots (49)$$

where

$$F(\eta) = \begin{bmatrix} f_1(\zeta, y) \\ f_2(\zeta, y) + g(\zeta, y)\alpha(\zeta, y) \end{bmatrix},$$

$$P(\eta)=\begin{bmatrix} p_1(\zeta,y) \\ p_2(\zeta,y)\end{bmatrix}, G(\eta)=\begin{bmatrix} 0 \\ g(\zeta,y)\end{bmatrix},$$

which has similar structure to (15). Hence, by Lemma 2, we can prove this theorem by constructing such a storage function $U(\eta)$ that the condition (21) is satisfied for the nominal system. We now consider the storage function of the system as follows

$$U(\eta)=\frac{1}{2}\left(U_1(\zeta)+\frac{1}{2}y^2\right) \quad\cdots\cdots\cdots\cdots\cdots\cdots (50)$$

Then,

$$\begin{aligned} &L_F U(\eta)+L_P U(\eta)w+L_G U(\eta)y_0 \\ &=\frac{1}{2}\frac{\partial U_1}{\partial\zeta}f_1(\zeta,y)+\frac{1}{2}y\alpha_1(\zeta,y)+\frac{1}{2}yg(\zeta,y)y_0 \\ &\quad+\frac{1}{2}\left[\frac{\partial U_1}{\partial\zeta}p_1(\zeta,y)+yp_2(\zeta,y)\right]w \\ &=\frac{1}{2}y\left\{\alpha_1(\zeta,y)+\frac{\partial U_1}{\partial\zeta}\tilde{f}_1(\zeta,y)+g(\zeta,y)y_0\right\} \\ &\quad+\frac{1}{2}\left\{\frac{\partial U_1}{\partial\zeta}f_1(\zeta,0)+\frac{\partial U_1}{\partial\zeta}p_1(\zeta,0)w\right\} \\ &\quad+\frac{1}{2}\left[\frac{\partial U_1}{\partial\zeta}\tilde{p_1}(\zeta,y)+p_2(\zeta,y)\right]yw \quad\cdots\cdots\cdots (51)\end{aligned}$$

Substituting (45) into (51) and completing the squares by adding and subtracting terms, we obtain

$$\begin{aligned} &L_F U(\eta)+L_P U(\eta)w+L_G U(\eta)y_0 \\ &\le\frac{1}{2}y\left\{\alpha_1(\zeta,y)+\frac{\partial U_1}{\partial\zeta}\tilde{f}_1(\zeta,y)+g(\zeta,y)y_0\right\} \\ &\quad-\frac{1}{2\gamma^2}\frac{\partial U_1}{\partial\zeta}p_1(\zeta,0)p_1^T(\zeta,0)\frac{\partial^T U_1}{\partial\zeta}-\frac{1}{2}\kappa_1(\|\zeta\|) \\ &\quad-\frac{1}{2}h^T(\zeta,0)h(\zeta,0)+\frac{1}{2}\frac{\partial U_1}{\partial\zeta}p_1(\zeta,0)w \\ &\quad+\frac{1}{2}\left[\frac{\partial U_1}{\partial\zeta}\tilde{p_1}(\zeta,y)+p_2(\zeta,y)\right]yw \\ &\le\frac{1}{2}y\left\{\alpha_1(\zeta,y)+\frac{\partial U_1}{\partial\zeta}\tilde{f}_1(\zeta,y)\right\}-\frac{1}{2}\kappa_1(\|\zeta\|) \\ &\quad+\frac{1}{2}yg(\zeta,y)y_0-\frac{1}{8\epsilon^2}g^2(\zeta,y)y^2+\frac{1}{8\epsilon^2}g^2(\zeta,y)y^2 \\ &\quad-\frac{1}{2\gamma^2}\frac{\partial U_1}{\partial\zeta}p_1(\zeta,0)p_1^T(\zeta,0)\frac{\partial^T U_1}{\partial\zeta}+\frac{1}{2}\frac{\partial U_1}{\partial\zeta}p_1(\zeta,0)w \\ &\quad-\frac{1}{2}h^T(\zeta,0)h(\zeta,0)-\frac{1}{2}H(\zeta,y)y+\frac{1}{2}H(\zeta,y)y \\ &\quad+\frac{1}{2}\frac{\partial U_1}{\partial\zeta}\tilde{p}_1(\zeta,y)yw-\frac{1}{\gamma^2}\frac{\partial U_1}{\partial\zeta}\tilde{p}_1(\zeta,y)\tilde{p}_1^T(\zeta,y)\frac{\partial^T U_1}{\partial\zeta}y^2 \\ &\quad+\frac{1}{\gamma^2}\frac{\partial U_1}{\partial\zeta}\tilde{p}_1(\zeta,y)\tilde{p}_1^T(\zeta,y)\frac{\partial^T U_1}{\partial\zeta}y^2+\frac{1}{\gamma^2}{p_2}^2(\zeta,y)y^2 \\ &\quad+\frac{1}{2}p_2(\zeta,y)yw-\frac{1}{\gamma^2}p_2^2(\zeta,y)y^2 \\ &\le\frac{1}{2}y\left\{\alpha_1(\zeta,y)+\frac{\partial U_1}{\partial\zeta}\tilde{f}_1(\zeta,y)+\frac{1}{4\epsilon^2}g^2(\zeta,y)y\right\} \\ &\quad+\frac{1}{2}y\left\{H(\zeta,y)+\frac{2}{\gamma^2}\frac{\partial U_1}{\partial\zeta}\tilde{p}_1(\zeta,y)\tilde{p}_1^T(\zeta,y)\frac{\partial^T U_1}{\partial\zeta}y\right\} \\ &\quad+\frac{1}{\gamma^2}p_2^2(\zeta,y)y^2-\frac{1}{2}h^T(\zeta,y)h(\zeta,y)-\frac{1}{2}\kappa_1(\|\zeta\|) \\ &\quad+\frac{\epsilon^2}{2}\|y_0\|^2+\frac{\gamma^2}{8}\|w\|^2+\frac{\gamma^2}{16}\|w\|^2+\frac{\gamma^2}{16}\|w\|^2\end{aligned}$$

Taking (47), (28) and (48) into account, we have

$$\begin{aligned} &L_F U(\eta)+L_P U(\eta)w+L_G U(\eta)y_0 \\ &\le\frac{\gamma^2}{4}\|w\|^2+\frac{\epsilon^2}{2}\|y_0\|^2-\frac{1}{2}\|z\|^2 \\ &\quad-\frac{1}{2}\left[\kappa_1(\|\zeta\|)+(\kappa_2(\|y\|)+k)y^2\right] \\ &\le\frac{\gamma^2}{4}\|w\|^2-\frac{1}{2}\|z\|^2+\nu(\|y_0\|)-[\kappa(2\|\zeta\|)+\kappa(2\|y\|)] \\ &\le\frac{1}{2}\left\{\frac{\gamma^2}{2}\|w\|^2-\|z\|^2\right\}+\nu(\|y_0\|)-\kappa(\|\eta\|)\end{aligned}$$

Therefore, by Lemma 2 we conclude that the closed-loop system satisfies $[P1]-[P3]$. ∎

Remark 3. In order to satisfy the condition (28), the parameter ϵ should be chosen as small as possible. And, to satisfy the condition (48) the parameter k should be chosen as large as possible. However, a small ϵ and a large k will make gain of the controller (30) and (46) higher. It is a trade-off to selecting suitable values for the parameters. Unfortunately, it should be noted that there is no general procedure in the trade-off.

Remark 4. Suppose that the nominal system is minimum-phase, i.e. the zero dynamics of the nominal system $\dot{\zeta}=f_1(\zeta,0)$ is globally asymptotically stable at $\zeta=0$, then, by inverse Lyapunov theorem, there exists a Lyapunov function $U_1(\zeta)$ satisfying $L_{f_1(\zeta,0)}U_1(\zeta)<0,\ \forall\zeta\neq 0$. Furthermore, if there exists a positive definite function $Q(\zeta)$ such that

$$L_{f_1(\zeta,0)}U_1(\zeta)<-Q(\zeta),\quad \forall\zeta \quad\cdots\cdots\cdots\cdots\cdots (52)$$

holds, and

$$h^T(\zeta,0)h(\zeta,0)+\kappa_1(\|\zeta\|)\le Q(\zeta),\quad\forall\zeta$$

is satisfied, then we have

$$L_{f_1(\zeta,0)}U_1(\zeta)+h^T(\zeta,0)h(\zeta,0)+\kappa_1(\|\zeta\|)\le 0,\ \forall\zeta \quad (53)$$

This is a special case of (45) when $p_1(\zeta,0)=0$. It means that if the zero dynamics is not forced directly by the disturbance w, and there exist a Lyapunov function $U_1(\zeta)$ and $Q(\zeta)$ satisfying (52) and (53), then a desired storage function satisfying the condition in Theorem 2 can be easily constructed based on the Lyapunov function $U_1(\zeta)$.

In fact, in the case where the zero dynamics is exponentially stable at $\zeta=0$, it is easy to find such a Lyapunov function U_1.

Remark 5. As is shown in Lemma 2, if the nominal system with dynamical uncertainty has the structure shown by (15), then a sufficient condition such that the closed-loop system satisfies $[P1]-[P3]$ can be given by a storage function satisfying the dissipative inequality (21) according to the nominal system. The result of Theorem 2 presents a step-by-step constructive way for the storage function when the nominal system has relative degree one. In fact, this recursive design method can be extended to more general system, which has relative degree $r>1$. For example, consider the nonlinear systems with the following form.

$$
\begin{cases}
\dot{\zeta} = f_1(\zeta, \eta_1) \\
\dot{\eta}_1 = \eta_2 + \phi_1(\zeta, \tilde{\eta}_1)w \\
\dot{\eta}_2 = \eta_3 + \phi_2(\zeta, \tilde{\eta}_2)w \\
\vdots \\
\dot{\eta}_r = f_2(\zeta, \tilde{\eta}_r) + \phi_r(\zeta, \tilde{\eta}_r)w + g(\zeta, \tilde{\eta}_r)v \\
y = \eta_1 \\
z = h(\zeta, \tilde{\eta}_r) \\
v = c(\xi) + u \\
\dot{\xi} = A(\xi) + bu
\end{cases} \tag{54}
$$

where $\tilde{\eta}_i^T = [\eta_1, \eta_2, \cdots, \eta_i]$ $(1 \le i \le r)$, $\zeta \in R^{n-r}$. $f_1(\zeta, \eta_1)$, $\phi_i(\zeta, \tilde{\eta}_i)$ and $h(\zeta, \tilde{\eta}_r)$ are smooth vector fields, $f_2(\zeta, \tilde{\eta}_r)$ and $g(\zeta, \tilde{\eta}_r)$ are smooth functions, $f_1(0,0) = 0$, $f_2(0,0) = 0$, $h(0,0) = 0$. If there exists a Lyapunov function $U_1(\zeta)$ satisfies the condition in Remark 4, the storage function can be constructed by recursive way based on the $U_1(\zeta)$.

As is well-known, if the nominal system (2) has relative degree $r > 1$, then under appropriate geometrical conditions, the system (1) is feedback equivalent to the system (54).

4. Numerical example

Consider a system given by

$$
\begin{cases}
\dot{x}_1 = -x_1^3 + x_2 + (x_1 + x_1 x_2)w \\
\dot{x}_2 = x_2 + 0.5x_1^2 + (1 + \sin(x_1 + x_2))w + (1 + x_1^2)v \\
z = \frac{1}{2}\begin{bmatrix} x_1^2 \\ x_2 \end{bmatrix} \\
\dot{\xi} = A(\xi) + bu \\
v = c(\xi) + u
\end{cases} \tag{55}
$$

It is easy to check that the nominal system has relative degree one. Let the disturbance attenuation level is given by $\gamma = 10$. We will design a state feedback controller for (55) such that the closed-loop system satisfies $[P1] - [P3]$ for all admissible uncertainty.

Using the notation of (39), $p_2(x_1, x_2) = 1 + \sin(x_1 + x_2)$, $g(x_1, x_2) = 1 + x_1^2$. Hence, it is easy to check $|p_2(x_1, x_2)| \le 2$, $g(x_1, x_2) \ge 1$, so that Assumption $[A2']$ is satisfied.

Choosing $\kappa_1(\|x_1\|) = 15x_1^4$, then $U_1(x_1) = 10x_1^2$ satisfies (45). According to Theorem 2 and choosing $\epsilon = 0.44$, $\kappa_2(\|x_2\|) = \frac{5}{2}x_2^2$, $k = 71.8476$, a desired state feedback controller is designed as

$$
u = \alpha(x_1, x_2) = \frac{1}{1 + x_1^2}[\alpha_1(x_1, x_2) - x_2 - \frac{1}{2}x_1^2] \tag{56}
$$

where

$$
\alpha_1(x_1, x_2) = -20x_1 - \frac{1}{50}x_2[400x_1^4 + (1 + \sin(x_1 + x_2))^2] - \frac{1}{4}x_2 - 1.2913(1 + x_1^2)^2 x_2 - (\frac{5}{2}x_2^2 + 71.8476)x_2
$$

Suppose the uncertainty is described by

$$
\begin{cases}
A(\xi) = \begin{bmatrix} -2\xi_1 - \xi_1^3 + \xi_2 \\ -\xi_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
c(\xi) = \xi_1 + \dfrac{-\xi_2 + 2\xi_2^3}{1 + \xi_2^2}
\end{cases} \tag{57}
$$

By the change of coordinates

$$
\begin{cases}
\bar{\xi}_1 = \xi_1 \\
\bar{\xi}_2 = \xi_2 - \int_0^{x_2} \frac{1}{1 + x_1^2} ds
\end{cases}
$$

the input uncertainty is transformed into the form (11), where

$$
d_1 = \begin{bmatrix} 0 \\ \dfrac{x_2}{1 + x_1^2} \end{bmatrix}
$$

$$
d_2 = \begin{bmatrix} 0 \\ -\dfrac{x_2 + 1/2x_1^2}{1 + x_1^2} - \dfrac{1 + \sin(x_1 + x_2)}{1 + x_1^2}w \end{bmatrix}
$$

$$
A_0(\xi) = \begin{bmatrix} -2\xi_1 - \xi_1^3 + \xi_2 \\ -\xi_2 - \xi_1 + \dfrac{\xi_2 - 2\xi_2^3}{1 + \xi_2{}^2} \end{bmatrix}
$$

If we construct a positive function $W(\bar{\xi}) = \frac{1}{2}\bar{\xi}_1^2 + \frac{1}{2}\bar{\xi}_2^2$, then, it is easy to show that the input uncertainty satisfies Assumption $[A1]$, where $\hat{c}(\|\xi\|) = 2\sqrt{2}\|\xi\|$, $\beta_1(\|\bar{\xi}\|) = \frac{1}{2}\|\bar{\xi}\|^2$, $\beta_2(\|d_1\|) = 7\|d_1\|^2$ and $\beta_3(\|d_2\|) = \|d_2\|^2$. Also, the storage function of the system (55) is

$$
V(x_1, x_2, \bar{\xi}_1, \bar{\xi}_2) = \frac{25}{8}W(\bar{\xi}) + \frac{1}{2}(U_1(x_1) + \frac{1}{2}x_2^2)
$$

Simulation results of the system (55) with the controller (56) and the uncertainty (57) are shown in Fig.1 and Fig.2. Fig.1 indicates that the closed-loop system is asymptotically stable when the initial state $x_1(0) = 1$, $x_2(0) = 1$ and disturbance input $w = 0$. While the disturbance input $w = 2\sin 3t$ and the initial state $x_1(0) = 0$, $x_2(0) = 0$, the boundedness of the states is demonstrated in Fig.2.

5. Conclusions

In this paper, we address the robust L_2 disturbance attenuation problem for nonlinear systems with input dynamical uncertainty. The uncertainty is restricted to be minimum-phase and relative degree zero. First, a sufficient condition is given based on a dissipation inequality such that the nonlinear systems satisfy L_2 gain performance and ISS property for all admissible uncertainty. The dissipation inequality is derived for the nominal system. Using this condition, a smooth state feedback control law is given, which solves the robust L_2 disturbance attenuation problem with ISS. Moreover, the design approach is extended to the case where the nominal system has higher relative degree by using the recursive method. Finally, a numerical example demonstrates the proposed approach.

(Manuscript received July 23, 2001, rivised Feb.12, 2002)

References

(1) A. J. van der Schaft, On a state space approach to nonlinear H_∞ control, *Systems and Control letters*, 16, pp.1-8, 1991.

(2) A. J. van der Schaft, L_2 gain analysis of nonlinear systems and nonlinear H_∞ control, *IEEE Trans. Automat. Contr.* Vol.37(6), pp.770-784, 1992.

(3) A. J. van der Schaft, L_2-gain and passivity techniques in nonlinear control, *springer*, London, 1995.

(4) A. Isidori, Nonlinear control systems, *springer*, London,1995.

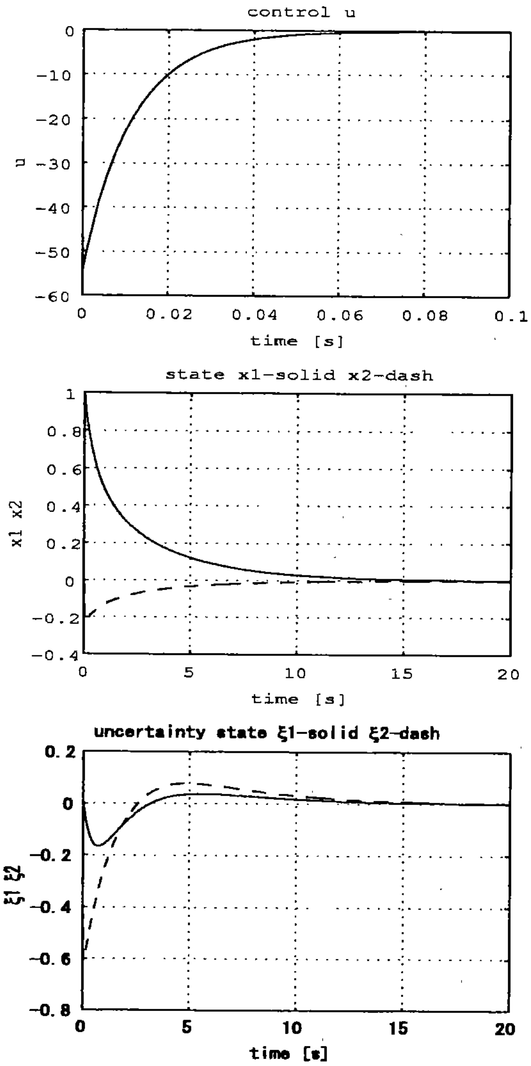


Fig. 1. $w = 0$, the state and control signal

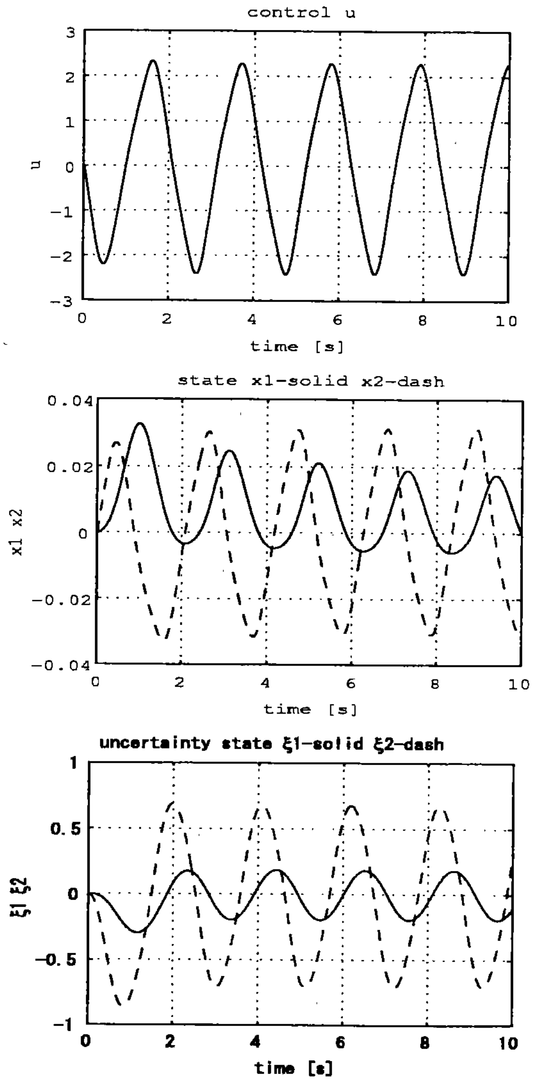


Fig. 2. $w = 2\sin 3t$, the state and control signal

(5) A. Isidori, Disturbance attenuation and H_∞ control via measurement feedback in nonlinear systems, *IEEE Trans. Automat. Contr.*, Vol.37(9), pp.1283-1293, 1992.

(6) A. Isidori, Global almost disturbance decoupling with stability for nonminimum-phase single-input single-output nonlinear systems, *Systems and Control letters*, 28, pp.115-122, 1996.

(7) E.D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. Automat. Contr.* Vol.34(4), pp.435-443, 1989.

(8) E.D. Sontag and Y. Wang, On characterizations of the input-to- state stability property, *Systems and Control letters*, 24, pp.351-359, 1995.

(9) L. Xie and W. Su, H_∞ control for a class of cascaded nonlinear systems, *IEEE Trans. Automat. Contr.* Vol.42(10), pp.1465-1469, 1997.

(10) M. Arcak and P. V. Kokotovic, Robust nonlinear control of systems with input unmoedeled dynamics, *Systems and Control letters*, 41, pp.115-122, 2000.

(11) R. Marino, W. Respondek, A. J. van der Schaft and P. Tomei, Nonlinear H_∞ almost disturbance decoupling, *Systems and Control letters*, 23, pp.159-168, 1994.

(12) R. Marino and P. Tomei, Nonlinear output feedback tracking with almost disturbance decoupling, *IEEE Trans. Automat. Contr.* Vol.44(1), pp.18-28, 1999.

(13) T. Shen and K. Tamura, Robust H_∞ control of uncertain nonlinear system via state feedback, *IEEE Trans. Automat. Contr.* Vol.40(4), pp.766-768, 1995.

(14) T. Shen and K. Tamura, Nonlinear robust H_∞ control- an approach based on Lyapunov function, *Transaction of SICE*, Vol.34(9), pp.1191-1197, 1998(in Japanese).

(15) T. Shen, L. Xie and K. Tamura, Robust almost disturbance decoupling for nonlinear systems with structural uncertainty, *Proc. 37th IEEE CDC*, Tampa, pp.4107-4108, 1998.

(16) T. Shen and K. Tamura, Robust feedback design of a class of nonlinear cascaded systems with structural uncertainty, *14th World Congress of IFAC*, Beijing, pp.447-451, 1999.

(17) T. Shen, L. Xie and K. Tamura, Constructive design approach to robust H_∞ control systems with gain bounded uncertainty, *Trans. of SICE* Vol.36(3), pp242-247, 2000.

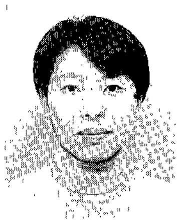
(18) W. Su, L. Xie and C.E. de Souza, Global robust disturbance attenuation and almost disturbance decoupling for uncertain cascaded nonlinear systems, *Automatica* Vol.35(4), pp.35-48, 1999.

(19) Z.P. Jiang and M. Arcak, Robust global stabilization with ignored input dynamics: an input-to-state (ISS) small-gain approach, *IEEE Trans. Automat. Contr.* Vol.46(9), pp.1411-1415, 2001.

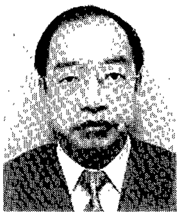
(20) Z.P. Jiang and M. Arcak, Robust global stabilization with input unmodeled dynamics: an ISS small-gain approach, *Proceedings of 39th IEEE Conference on Decision and Control*, pp.1301-1306, 2000.

(21) Z.P. Jiang, A. Teel and L. Praly, Small-gain theorem for ISS systems and applications, *Mathematics of Control, Signals and system*, Vol.7 pp.95-120, 1994.

Xiaohong Jiao (Non-member) Xiaohong Jiao received the B. Eng. and M. Sc. degrees in Automatic Control from Northeast Heavy Machinery Institute, China in 1988 and 1991, respectively. From April 1991 to October 2000, she served as a Teaching Assistant, a Lecture and then an Associate Professor in the Department of Automatic Control, Yanshan University, China. Now, she is studying the Ph. D course in Department of Mechanical Engineering at Sophia University. She is working on robust control and adaptive control of nonlinear systems.



Tielong Shen (Member) Tielong Shen received the B. Eng. and M. Sc. degrees in Automatic Control from Northeast Heavy Machinery Institute, China in 1982 and 1986, respectively, and the Ph. D.degree in Mechanical Engineering from Sophia University, Tokyo, Japan, 1992. From March 1986 to March 1989, he served as a teaching assistant and then a lecture in the Department of Automatic Control, Northeast Heavy Machinery Institute. Since April 1992, he has been a faculty member as an Assistant Professor of Department of Mechanical Engineering at Sophia University. He is the author/coauthor of three textbooks. His current research interests include H_∞ control theory, robust control of linear and nonlinear systems and its applications in mechanical systems.



Katsutoshi Tamura (Non-member) After receiving Master Degree from Nagoya University, he was appointed to a research associate there, and began his Ph. D research on optimal control theory and algorithm to obtain optimal control input to linear and nonlinear systems. After this he moved to Sophia University as a Lecturer and got promotion to an Associate professor and to a Professor. Now, he is working on adaptive control and nonlinear control.