Paper

Satisfactory Efficient Linear Coordination Method for Multi-Objective Linear Programming Problems with Convex Polyhedral Preference Functions

Busaba Phruksaphanrat*, Non-member, Ario Ohsato**, Member

At present, the most commonly used satisficing method for multi-objective linear programming (MOLP) is the goal programming (GP) based method but this method does not always generate efficient solutions. Recently, an efficient GP-based method, which is called reference goal programming (RGP), has been proposed. However, it is limited to only a certain target point preference, which is too rigid. More flexible preferences such convex polyhedral preferences are preferred for many practical problems. In this research, a satisfactory effective linear coordination method for MOLP problems with convex polyhedral preference functions is proposed. The concept of the convex cone is used to formulate the convex polyhedral preference function and the existing lexicographic model of the reference point method (RPM) is integrated to ensure the efficiency of the solution of the problem. The formulated model can be solved by existing linear programming solvers and can find the satisfactory efficient solution. The convex polyhedral function enriches the existing preferences for efficient methods and increases the flexibility in designing preferences for decision makers.

In some situation, it is difficult for a decision maker to state a certain desirable level for each objective function. Applying fuzzy goal to capture the decision maker's preferences has the advantage of allowing for vague aspirations, which can be considered as convex polyhedral preference functions. The satisfactory efficient linear coordination method can be applied to obtain an efficient solution, which is also close to the decision maker's requirements.

Keywords : convex cone, multi-objective problem, goal programming, reference point method, preference function, and efficient solution

1. Introduction

Fundamental to a multi-objective linear programming (MOLP) problem is the *Pareto optimal concept*, which is also known as an *efficient solution* or a *nondominated solution*. The efficient solution of the MOLP problem is one where any improvement of one objective function can be achieved only at the expense of another [(1)].

All of the efficient solutions set consists of points or elements having a simple and highly desirable property [(1)]. However, the efficient solutions of real-world problems are extremely numerous and almost impossible to compare [(2)]. So, the need to consider additional information from the decision maker arises. A preference function can be used by the decision maker to reduce the efficient solution set. Normally, the preference modeling technique is applied to goal programming (GP) based methods [(3)~(10)], which are multi-objective programming techniques using satisficing concept [(11)(12)]. Satisficing solutions may not be the efficient solutions. This is a serious flaw of GP-based methods.

Recently, reference goal programming (RGP) [(13)~(17)] has been proposed for finding the efficient solution of an MOLP problem with certain target point preferences. This method expresses the reference point method (RPM) [(11)(12)(16)] by GP. It always guarantees the efficiency of the solution, which is different from typical GP formulations (weighted goal programming (WGP) and minmax GP), which do not. However, only a certain target point preference is considered, so this method is too rigid. Interval preference structures, increasing in preferences, decreasing in preferences, varying preference levels depended on the distance from the goals or other piecewise linear preferences are more flexible [(2)]. These kinds of preference structures can be collectively called *convex polyhedral type* [(25)] preference functions, which are nonlinear preference functions. GP with convex polyhedral preference functions have been used for many practical problems such as financial planning, reservoir operation, manufacturing systems, etc [(7)(18)~(24)]. However, it is only possible to find satisficing solutions, which are not always efficient [(14)(15)(26)]. This research developed the satisfactory efficient method for an MOLP problem to overcome this weakness. The convex cone concept for convex polyhedral functions [(25)] and the existing lexicographic RPM are applied to develop the satisfactory efficient method, which always ensures the efficiency of the solution and satisfies the decision maker's requirements. For each time of formulation, a single solution that dominates the GP solution is produced.

For imprecise knowledge, normally fuzzy set theory has been applied [(18)(29)~(32)] to express goals or preferences for objectives. A fuzzy goal [(33)] and linguistic information [(34)(35)] can be used to express an aspiration level and a concave polyhedral membership function for allowing decision makers to represent their information in a more direct way when he/she is unable to express it precisely. The existing efficient method, the augmented minmax method, has been proposed to find the efficient solution for continuous strictly monotone decreasing membership functions and continuous strictly monotone increasing membership functions [(36)(37)]. The more convenient efficient method for flexible membership functions such as concave polyhedral membership

* Graduate school of Engineering, Nagaoka University of Technology
1603-1, Kami-Tomioka, Nagaoka, Niigata 940-2188
** Department of Management and Information Systems Science, Nagaoka University of Technology
1603-1, Kami-Tomioka, Nagaoka, Niigata 940-2188

functions needs to be developed because the concave polyhedral membership function can be used to represent a linear, a triangular, or a trapezoidal membership function, and it is also possible to use it as an approximation membership function of a concave nonlinear membership function. Our satisfactory efficient linear coordination method can also be applied to concave polyhedral membership functions. The resulting solution is efficient and also close to the decision maker's requirements.

The remainder of this paper is divided into 3 sections. Section 2 discusses the problem model and existing methods in detail. Section 3 gives the formulation of the satisfactory efficient method. To illustrate the application of the method, examples for both non-fuzzy and fuzzy situations are also included in section 4 of this paper.

2. Lexicographic Models

Consider an MOLP problem [1][38] with K objective functions. The general model can be formulated as follow:

$$\min\{[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_K(\mathbf{x})] : \mathbf{x} \in Q\}, \quad \cdots\cdots (1)$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected with in the feasible set Q; $\mathrm{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$, and $f_k(\mathbf{x})$ is the kth objective function, $k = 1,2,\ldots,K$.

The efficient solution (a nondominated solution), $[f_1(\bar{\mathbf{x}}), f_2(\bar{\mathbf{x}}), \ldots f_K(\bar{\mathbf{x}})]$ is defined to stand for the following inequalities

$$f_k(\bar{\mathbf{x}}) \leq f_k(\mathbf{x}), \text{ for } k = 1,2,\ldots,K,$$

$$\text{and } f_k(\bar{\mathbf{x}}) < f_k(\mathbf{x}) \text{ for at least one } k,\ \mathbf{x} \in Q \quad \cdots\cdots (2)$$

RPM [11][12][39] is the technique, underlying optimizing philosophy [6], where the decision maker specifies preferences in terms of reference levels. A scalarizing achievement function is built, corresponding to the specified reference levels. It generates an efficient solution to the problem. The generic scalarizing achievement function can be lexicographically considered as the problem in Eq.(3) [14]. This equation means that the first achievement function, $\max_{1\le k\le K}\{g_k(l_k, y_k)\}$ is minimized and then within the set of optimal solutions to the first function the second function, $\sum_{k=1}^{K} g_k(l_k, y_k)$ is minimized.

$$\text{lex min}\left\{\left[\max_{1\le k\le K}\{g_k(l_k,y_k)\}, \sum_{k=1}^{K} g_k(l_k,y_k)\right] : \mathbf{x} \in Q\right\}, \quad \cdots\cdots (3)$$

where y_k denotes the mathematical expression of the kth objective, ($y_k = f_k(\mathbf{x})$), l_k denotes reference levels, and $g_k : R^2 \rightarrow R$, for $k = 1,2,\ldots,K$, are the individual achievement functions measuring actual achievement of the kth objective with respect to the corresponding reference level, l_k. The advantage of the above lexicographic model is that it allows the decision maker to generate all efficient solutions. The simplest model of g_k can be modeled as a two segment piecewise linear function:

$$g_k(l_k, y_k) = \begin{cases} s_k^-(l_k - y_k), & \text{if } y_k \le l_k, \\ s_k^+(y_k - l_k), & \text{otherwise,} \end{cases} \quad \cdots\cdots (4)$$

where s_k^- and s_k^+ are the negative and positive weights corresponding to under-achievement and over-achievement, respectively.

The optimal set of the minmax aggregation alone (the first priority in Eq. (3)) always contains the efficient solution. Thus, if unique, the optimal solution of the minmax aggregation is efficient. In the case of a multi-objective optimization, one of them is efficient but some of them may not be efficient (in the case that some goals are set too pessimistically [6]). This is a serious flaw since practical problems usually have multiple optimal solutions. To overcome this flaw, it is regularized with the weighted aggregation (the second priority in Eq. (3)) to guarantee the efficiency of the solution [15].

Under the assumption that s_k^- is much smaller than s_k^+, Eq. (3) can be expressed in terms of GP implementation environment as the following lexicographic model, called reference goal programming (RGP) [13][15]:

$$\text{lex min}\left\{\left[\max_{1\le k\le K}\{-s_k^- n_k + s_k^+ p_k\}, \sum_{k=1}^{K}(-s_k^- n_k + s_k^+ p_k)\right]\right\}, \quad \cdots\cdots (5)$$

$$\text{subject to } f_k(\mathbf{x}) + n_k - p_k = r_k,\ \mathbf{x}, n_k, p_k \ge 0,\ \mathbf{x} \in Q,$$

where the aspiration level, r_k, is also employed as the reference level, l_k, and n_k and p_k represent the kth negative and positive deviational variables.

If we set r_k equal to the ideal value of the corresponding k objective for all k, then the efficiency of the solution provided by the first priority of lexicographic model in Eq. (5) is guaranteed and the regularization term becomes unnecessary [6].

Ogryczak [15] has redeemed Eq. (5), which is limited to the assumption that $0 < s_k^- \le s_k^+$ for k = 1, 2, ..., K, by adding more priority levels of the lexicographic RGP model as follows:

$$\text{lex min}\left\{\left[\max_{1\le k\le K}\{s_k^+ p_k\}, \sum_{k=1}^{K}(s_k^+ p_k), \max_{1\le k\le K}\{-s_k^- n_k\}, \sum_{k=1}^{K}(-s_k^- n_k),\right]\right\}, \quad \cdots\cdots (6)$$

$$\text{subject to } f_k(\mathbf{x}) + n_k - p_k = r_k,\ \mathbf{x}, n_k, p_k \ge 0,\ \mathbf{x} \in Q,$$

where s_k^- and s_k^+ represent freely selected negative and positive weights corresponding to under-achievement and over-achievement, respectively.

The corresponding RGP model always guarantees the efficiency of the solution. However, only a certain target point preference can be used. Various preference functions, g_k provide a wide modeling environment for measuring individual achievements so the convex polyhedral preference function as shown in Fig.1, which is the collective model for many types of preference functions is determined.

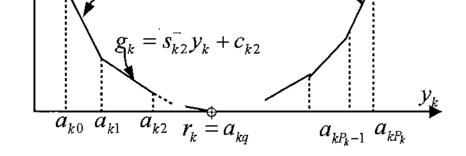

Fig. 1. The convex polyhedral preference function of the kth objective.

This convex polyhedral preference function can be mathematically represented by

$$g_k(r_k,y_k)=\begin{cases} s_{k1}^- y_k + c_{k1}, & \text{if } a_{k0}\le y_k < a_{k1},\\ s_{k2}^- y_k + c_{k2}, & \text{if } a_{k1}\le y_k < a_{k2},\\ \vdots & \\ s_{kq}^- y_k + c_{kq}, & \text{if } a_{kq-1}\le y_k < a_{kq}\text{ ; where } r_k = a_{kq},\\ s_{kq+1}^+ y_k + c_{kq+1}, & \text{if } a_{kq}\le y_k < a_{kq+1}\\ \vdots & \\ s_{kP_k}^+ y_k + c_{kP_k}, & \text{if } a_{kP_{k-1}}\le y_k < a_{kP_k}, \end{cases} \quad \cdots\cdots(7)$$

where

r_k is the aspiration level which is equal to the reference level, l_k,
a_{kd} is the dth breakpoint of $g_k(r_k,y_k)$, $d = 0, 1, \ldots, P_k$, $k = 1, 2,\ldots, K$,
s_{kd}^- and s_{kd}^+ are the corresponding slopes of the line segment in the range (a_{kd-1},a_{kd}) of the negative and the positive sides of r_k, $d = 1, 2,\ldots, P_k$, $k = 1,\ldots, K$,
c_{kd} is the y-intercept of the corresponding line segment, $d = 1, 2,\ldots, P_k$, $k = 1,\ldots, K$,
s_{kq}^-, c_{kq} and s_{kq+1}^+, c_{kq+1} are the corresponding slope and the y-intercept of the line segment of the negative and the positive sides of a_{kq}.

Generally, the GP for the convex polyhedral preference function can be applied by determining each breakpoint as a goal. For a triangular type preference function, it is not so difficult to combine GP with the RPM, but it is difficult to use only one reference point to model a convex polyhedral preference function, which means that it is difficult to extend the flexibility of designing preference functions by GP.

In the next section, the satisfactory efficient linear coordination method based on the convex cone concept is developed as an alternative to GP. By integrating the convex cone concept with the lexicographic RPM, one reference point is sufficient to formulate MOLP problems with convex polyhedral preference functions.

3. Satisfactory Efficient Linear Coordination Method for Multi-Objective Linear Programming Problems with Convex Polyhedral Preference Functions

In this paper, we propose a satisfactory efficient linear coordination method for solving MOLP problems with convex polyhedral preference functions. The concepts of convex cone and the lexicographic RPM are applied.

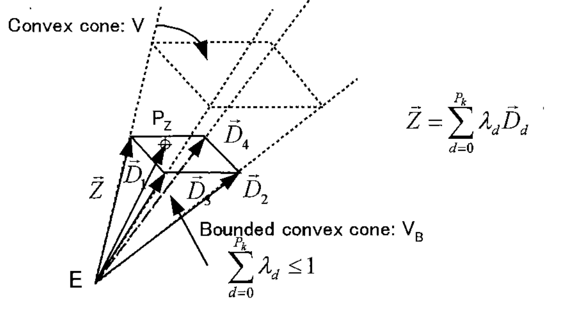

Fig. 2. Convex cone V and bounded convex cone V_B.

3.1 The Concept of Convex Cone From the concept of convex cone, it is possible to find any vector $\vec{Z}$ in the convex cone V by the following equation:

$$V=\left\{\vec{Z}\,\middle|\,\vec{Z}=\sum_{d=0}^{P_k}\lambda_d\vec{D}_d,\lambda_d\ge 0, d=0,1,\ldots,P_k\right\}, \quad \cdots\cdots(8)$$

where $\vec{D}_d$ are vectors from an extreme point, E to some points on the specified space and λ_d are coefficients related to each $\vec{D}_d$. With the additional constraint,

$$\sum_{d=0}^{P_k}\lambda_d \le 1, \quad \cdots\cdots(9)$$

a bounded convex cone V_B can be formed within the convex cone V. This bounded convex cone is a *convex polyhedral* [25].

The convex cone concept in Eq. (8) and Eq. (9) can be used to form convex polyhedral preference functions in MOLP problems by linear functions so it can be called *a linear coordination method*.

3.2 Formulations of Linear Doordination Method Based on the Convex Cone Concept In this section, an MOLP problem with convex polyhedral preference functions is formulated by applying the convex cone concept to form K convex polyhedral cones of K preference functions. The convex polyhedral cone of the kth preference function is shown in two-dimensional space in Fig.3. Horizontal axis represents the kth objective value and vertical axis represents the dissatisfaction level. The detail discussion of the formulations by the convex cone concept is explained in Ref.25. Given:

E_k : the most desirable point of the kth objective function, $E_k = (r_k, 0)$, $k = 1, 2,\ldots,K$,
$\vec{E}_k$: the vector from the origin (0,0) to point E_k, $\vec{E}_k=[r_k,0]^T$,
B_{kd}: the dth breakpoint of $g_k(r_k,y_k)$, $B_{kd} = (a_{kd}, g'_{kd})$, $k = 1, 2,\ldots,K$, $d = 0, 1, \ldots, P_k$,
$\vec{B}_{kd}$: the vector from the origin (0,0) to point B_{kd}, $\vec{B}_{kd}=[a_{kd},g'_{kd}]^T$,
g'_{kd}: the normalized value of function $g_k(r_k,y_k)$ at B_{kd}, which shows the level of dissatisfaction , $k = 1, 2, \ldots,K$, $d = 0, 1, \ldots, P_k$,
B_{kq} : the most desirable breakpoint $B_{kq} = E_k$, $k = 1, 2,\ldots,K$,
b_{kd}^- and b_{kd}^+ : the deviational constants for the negative and positive sides of the aspiration level, r_k on y_k axis; $d = 0, 1, \ldots, q$ for b_{kd}^- and $d = q+1, \ldots,P_k$ for b_{kd}^+, $k = 1, 2, \ldots, K$,
P_z^k : a point in the bounded convex polyhedral cone, $k = 1, 2,\ldots,K$,
$\vec{P}_z^k$: the vector from the origin (0,0) to point P_z^k,
$\vec{Z}$: the vector from E_k to P_z^k, and
$\vec{D}_{kd}$: the vector from E_k to the breakpoint B_{kd}, $k = 1, 2,\ldots,K$, $d = 0$,

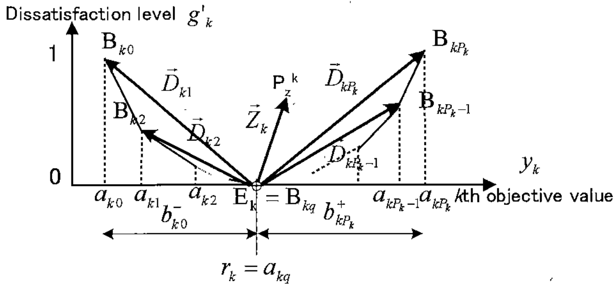

Fig. 3. A convex polyhedral cone.

1, ..., P_k.

$$\vec{D}_{kd} = \vec{B}_{kd} - \vec{E}_k,\ d = 0,1,...,P_k, \qquad (10)$$

To find P_z^k, V_B^k the kth bounded convex cone of the normalized function, $g'_k(r_k, y_k)$, can be constructed as

$$V_B^k = \left\{ \vec{Z}_k \mid \vec{Z}_k = \sum_{d=0}^{P_k} \lambda_{kd}\vec{D}_{kd}, \sum_{d=0}^{P_k} \lambda_{kd} \le 1, \lambda_{kd} \ge 0, d = 0,1,...,P_k \right\}, \qquad (11)$$

In Eq.(11), the convex cone concept in Eq.(8) and the additional constraint Eq.(9) for bounding the convex cone are applied to form a convex polyhedral preference function for the kth objective.

$$\vec{Z}_k = \begin{bmatrix} -\sum_{d=0}^{q} b_{kd}^- \lambda_{kd} + \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} \\ \sum_{d=0}^{P_k} g'_{kd} \lambda_{kd} \end{bmatrix}, \qquad (12)$$

and, $\vec{Z}_k = \vec{P}_z^{\,k} - \vec{E}_k$. $\qquad (13)$

Then, the position vector of $\vec{P}_z^{\,k}$ becomes

$$\vec{P}_z^{\,k} = \vec{E}_k + \vec{Z}_k = \begin{bmatrix} p_z^{\,k} \\ g'(p_z^{\,k}) \end{bmatrix} = \begin{bmatrix} r_k - \sum_{d=0}^{q} b_{kd}^- \lambda_{kd} + \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} \\ \sum_{d=0}^{P_k} g'_{kd} \lambda_{kd} \end{bmatrix} \qquad (14)$$

$p_z^{\,k}$ is the value of y_k, which is the constraint of the problem and $g'(p_z^{\,k})$ is the normalized value of $g_k(r_k, y_k)$ at $y_k = p_z^{\,k}$, which is the objective value of the problem.

The formulation of a convex polyhedral preference function for a single objective function problem of the linear coordination method can be shown as

$$\min \sum_{d=0}^{P_k} g'_{kd} \lambda_{kd}, \qquad (15)$$

$$\text{subject to } f_k(\mathbf{x}) + \sum_{d=0}^{q} b_{kd}^- \lambda_{kd} - \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} = r_k,$$

$$\sum_{d=0}^{P_k} \lambda_{kd} \le 1,$$

$$\mathbf{x} \in Q,\ \mathbf{x}, \lambda_{kd} \ge 0,\ d = 0,1,\ldots, P_k.$$

For multi-objective problems, an additive model [40] and a minmax model can be developed based on the function of the single objective problem.

The additive model of the linear coordination method for an MOLP problem with convex polyhedral preference functions based on the convex cone concept (LCC) can be shown as

$$\text{LCC: } \min \sum_{k=1}^{K} \sum_{d=0}^{P_k} g'_{kd} \lambda_{kd}, \qquad (16)$$

subject to

$$f_k(\mathbf{x}) + \sum_{d=0}^{q} b_{kd}^- \lambda_{kd} - \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} = r_k, k = 1,2,..., K, \qquad (17)$$

$$\sum_{d=0}^{P_k} \lambda_{kd} \le 1, k = 1,2,..., K, \qquad (18)$$

$$\mathbf{x} \in Q;\ \mathbf{x},\ \lambda_{kd} \ge 0,\ k = 1,\ldots, K,\ d = 0,1,\ldots,P_k. \qquad (19)$$

The minmax model of the linear coordination method for an MOLP problem based on the convex cone concept (minmax CC) can be modeled as

$$\text{minmax CC: } \min \alpha, \qquad (20)$$

$$\text{subject to } \alpha \ge \sum_{d=0}^{P_k} g'_{kd} \lambda_{kd},\ k = 1,\ldots, K, \qquad (21)$$

and Eq. (17)- Eq. (19).

where α denotes a new variable.

Both of the additive and the minmax model can be applied. However, they are based on the satisficing concept, which cannot ensure the efficiency of solutions. In order to improve these methods, the RPM concept is employed.

3.3 Satisfactory Efficient lLnear Coordination Method Based on the Convex Cone Concept (ECC) Using the RPM formulation, which always generates the efficient solutions to an MOLP problem with certain target point preferences, and the convex cone concept, the satisfactory efficient linear coordination method for convex polyhedral type preference functions can be formulated as follows.

From Eq. (17), we have

$$f_k(\mathbf{x}) + \sum_{d=0}^{q} b_{kd}^- \lambda_{kd} - \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} = r_k, k = 1,2,..., K.$$

$g_k(l_k, y_k)$ of the lexicographic RPM formulation stated in Eq. (3) can be considered as,

$$g_k(l_k, y_k) = (f_k(\mathbf{x}) - r_k) = -\sum_{d=0}^{q} b_{kd}^- \lambda_{kd} + \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd}, \qquad (22)$$

with the assumption that, $\sum_{d=0}^{q} b_{kd}^- \lambda_{kd} \le \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd}$. Then, the lexicographic model based on the convex cone concept can be formulated as

$$\text{lexmin}\left[\max_{1 \le k \le K} \left(-\sum_{d=0}^{q} g'_{kd} \lambda_{kd} + \sum_{d=q+1}^{P_k} g'_{kd} \lambda_{kd}\right), \sum_{k=1}^{K} \left(-\sum_{d=0}^{q} g'_{kd} \lambda_{kd} + \sum_{d=q+1}^{P_k} g'_{kd} \lambda_{kd}\right), \right] \qquad (23)$$

subject to Eq. (17)- Eq. (19).

This formulation can be used to find the satisfactory efficient solution for MOLP problems with convex polyhedral preference functions when the assumption, $\sum_{d=0}^{q} b_{kd}^- \lambda_{kd} \le \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd}$ is satisfied.

Under the constraint, $\sum_{d=0}^{q} b_{kd}^- \lambda_{kd} \times \sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd} = 0$ for all k, it can ensure that $\sum_{d=0}^{q} b_{kd}^- \lambda_{kd}$ and $\sum_{d=q+1}^{P_k} b_{kd}^+ \lambda_{kd}$ do not occur at the same time for all k. This constraint does not need to be added to the formulation model because all other constraints are linear [17]. The problem in Eq.(23) can be solved using a modified simplex method where b_{kd}^- and b_{kd}^+ are not selected as basic variables simultaneously. It is similar to the GP formulation that we could be sure that in every iteration this kind of constraint would be satisfied [8]. Only the boundary solutions of the convex polyhedral cone are considered. Then, the satisfactory efficient linear coordination method based on the convex cone concept (ECC) for convex polyhedral preference functions can be formulated as lex min

$$\left[\max_{1\le k\le K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\lambda_{kd}\right),\sum_{k=1}^{K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\lambda_{kd}\right),\max_{1\le k\le K}\left(-\sum_{d=0}^{q} g'_{kd}\lambda_{kd}\right),\sum_{k=1}^{K}\left(-\sum_{d=0}^{q} g'_{kd}\lambda_{kd}\right)\right] \quad \cdots\cdots (24)$$

subject to

$$f_k(\mathbf{x})+\sum_{d=0}^{q} b^-_{kd}\lambda_{kd}-\sum_{d=q+1}^{P_k} b^+_{kd}\lambda_{kd}=r_k, k=1,2,\ldots,K,$$

$$\sum_{d=0}^{P_k}\lambda_{kd}\le 1, k=1,2,\ldots,K,$$

$$\mathbf{x}\in Q;\ \mathbf{x},\ \lambda_{kd}\ge 0,\ k=1,\ldots,K,\ d=0,1,\ldots,P_k.$$

If we set r_k equal to the ideal value of the corresponding k objective for all k, then the regularization terms in the model Eq. (24) are not necessary to guarantee the efficiency of the solution. The model can be reduced to two priority levels.

The model in Eq. (24) can be used to find the efficient solutions of MOLP problems with nonlinear preferences, which is more advantageous than existing methods because it can generate the efficient solutions (Theorem), which is close to the decision maker requirements. The flexibility of this preference function enhances the decision analysis of decision makers in designing the objective's preference functions.

Theorem

For any reference level r_k and any positive value of deviational constants, b^-_{kd}, b^+_{kd}, if $(\bar{\mathbf{x}},\bar{\boldsymbol{\lambda}})$ is an optimal solution to the problem of ECC, then $\bar{\mathbf{x}}$ is an efficient solution to the multi-objective optimization Eq. (1) with preference functions in Eq. (7).

Proof

Let $(\bar{\mathbf{x}},\bar{\boldsymbol{\lambda}})$ be an optimal solution to the problem of ECC. Suppose that $\bar{\mathbf{x}}$ is not efficient to the problem in Eq. (2) with preference functions as in Eq. (7). It means that there exists a vector $\mathbf{x}\in Q$ such that

$$f_k(\mathbf{x})\le f_k(\bar{\mathbf{x}}) \text{ for all } k=1,2,\ldots,K, \quad \cdots\cdots (25)$$

and for some j $(1\le j\le K)$,

$$f_k(\mathbf{x})< f_k(\bar{\mathbf{x}}),$$

in other terms,

$$\sum_{k=1}^{K} f_k(\mathbf{x})<\sum_{k=1}^{K} f_k(\bar{\mathbf{x}}) \quad \cdots\cdots (26)$$

From the ECC formulation, we have

$$f_k(\bar{\mathbf{x}})-r_k=\sum_{d=q+1}^{P_k} b^+_{kd}\bar{\lambda}_{kd}-\sum_{d=0}^{q} b^-_{kd}\bar{\lambda}_{kd}, \quad \cdots\cdots (27)$$

where, $\sum_{d=0}^{P_k}\lambda_{kd}\le 1, k=1,2,\ldots,K.$

The solution of the problem cannot exist in both of the negative and the positive sides of r_k at the same time. Then, we have

$$\sum_{d=0}^{q} b^-_{kd}\bar{\lambda}_{kd}\times\sum_{d=q+1}^{P_k} b^+_{kd}\bar{\lambda}_{kd}=0, \quad \cdots\cdots (28)$$

$$f_k(\bar{\mathbf{x}})-r_k=\sum_{d=q+1}^{P_k} b^+_{kd}\bar{\lambda}_{kd},\qquad r_k-f_k(\bar{\mathbf{x}})=\sum_{d=0}^{q} b^-_{kd}\bar{\lambda}_{kd}, \quad \cdots\cdots (29)$$

where $b^+_{kd}, b^-_{kd}\ge 0$.

By the same way,

$$f_k(\mathbf{x})-r_k=\sum_{d=q+1}^{P_k} b^+_{kd}\lambda_{kd},\qquad r_k-f_k(\mathbf{x})=\sum_{d=0}^{q} b^-_{kd}\lambda_{kd}. \quad \cdots\cdots (30)$$

for all $k=1,2,\ldots,K$.

$(\mathbf{x},\boldsymbol{\lambda})$ is a feasible solution to the problem of ECC and due to Eq. (29) and Eq. (30) for any objective value with the convex polyhedral preference function related to $\boldsymbol{\lambda}$, the following inequalities are satisfied:

$$\sum_{d=q+1}^{P_k} g'_{kd}\lambda_{kd} \le \sum_{d=q+1}^{P_k} g'_{kd}\bar{\lambda}_{kd} \text{ for all } k=1,2,\ldots,K.$$

$$-\sum_{d=0}^{q} g'_{kd}\lambda_{kd} \le -\sum_{d=0}^{q} g'_{kd}\bar{\lambda}_{kd} \text{ for all } k=1,2,\ldots,K$$

Then,

$$\max_{1\le k\le K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\lambda_{kd}\right) \le \max_{1\le k\le K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\bar{\lambda}_{kd}\right),$$

$$\max_{1\le k\le K}\left(-\sum_{d=0}^{q} g'_{kd}\lambda_{kd}\right) \le \max_{1\le k\le K}\left(-\sum_{d=0}^{q} g'_{kd}\bar{\lambda}_{kd}\right),$$

and

$$\sum_{k=1}^{K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\lambda_{kd}\right) < \sum_{k=1}^{K}\left(\sum_{d=q+1}^{P_k} g'_{kd}\bar{\lambda}_{kd}\right),$$

$$\sum_{k=1}^{K}\left(-\sum_{d=0}^{q} g'_{kd}\lambda_{kd}\right) < \sum_{k=1}^{K}\left(-\sum_{d=0}^{q} g'_{kd}\bar{\lambda}_{kd}\right),$$

which contradicts optimality in Eq. (2) of $(\bar{\mathbf{x}},\bar{\boldsymbol{\lambda}})$ for the problem of ECC. Thus, $\bar{\mathbf{x}}$ must be the efficient solution to the multi-objective optimization Eq. (1) with preference functions in Eq. (7). □

Note that the theorem assumed $\sum_{d=0}^{q} b^-_{kd}\bar{\lambda}_{kd}\times\sum_{d=q+1}^{P_k} b^+_{kd}\bar{\lambda}_{kd}=0$. We directly put this assumption, which is used to guarantee the proper calculation of all deviations, into the problem model by considering each of the linear coordination of negative vectors and positive vectors from the reference level lexicographically. The constraints related to this assumption can be simply omitted from the problem.

In the case of imprecise knowledge, fuzzy approach is applied to represent the aspiration level with respect to the linguistic information from the decision maker, which is represented by a membership function. The discussion about membership functions is included in Appendix A. A concave polyhedral membership function can be considered. Then, the formulation in Eq. (24) can be used to solve these fuzzy optimization problems.

4. Numerical examples

In the previous section, we have introduced the satisfactory efficient linear coordination method based on the convex cone concept, which can solve MOLP problems with convex polyhedral preference functions. In this section, two cases of an MOLP problem are determined: A non-fuzzy MOLP problem and a fuzzy MOLP problem.

4.1 A Non-Fuzzy MOLP Problem with Convex Polyhedral Preference Functions In order to illustrate effectiveness of the proposed model, we compare their results with existing models, GP-based methods (Jones and Tamiz' s formulations [22]) on a small example of an MOLP problem.

Consider the following optimization problem with two objective functions with equal weights of their importance:

$$\min f_1(\mathbf{x}) = x_1, \quad \cdots\cdots (31)$$

$$\min f_2(\mathbf{x}) = x_2, \quad \cdots\cdots (32)$$

$$\text{subject to } 3x_1 + 4x_2 \geq 30, \quad \cdots\cdots (33)$$

$$x_1 \geq 2, \quad \cdots\cdots (34)$$

$$x_2 \geq 3. \quad \cdots\cdots (35)$$

The preference functions of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ have been defined and shown in the following figures.

Let l_1 and l_2 represent the reference levels of objective 1 and objective 2 accordingly.

The efficient set of this problem is

$$3x_1 + 4x_2 \geq 30,\ x_1 \geq 2,\ x_2 \geq 3,$$

i.e. the entire line segment between vertices (2,6) and (6,3), including vertices.

The WGP model for this MOLP problem becomes

$$\min\ n_2 + 2p_3 + \frac{1}{4}n_6 + \frac{1}{8}n_7 + \frac{1}{8}p_7 + \frac{1}{4}p_8, \quad \cdots\cdots (36)$$

$$\text{subject to } x_1 + n_2 - p_2 = l_1, \quad \cdots\cdots (37)$$

$$x_1 + n_3 - p_3 = l_1 + 2, \quad \cdots\cdots (38)$$

$$x_2 + n_6 - p_6 = l_2 - 4, \quad \cdots\cdots (39)$$

$$x_2 + n_7 - p_7 = l_2, \quad \cdots\cdots (40)$$

$$x_2 + n_8 - p_8 = l_2 + 4, \quad \cdots\cdots (41)$$

Eq. (33)- Eq. (35),

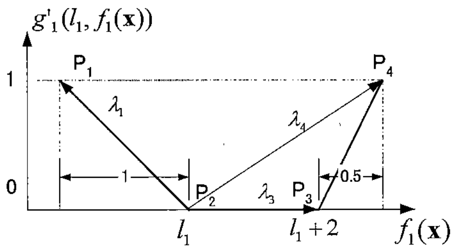

Fig. 4. Preference function of $f_1(\mathbf{x})$.

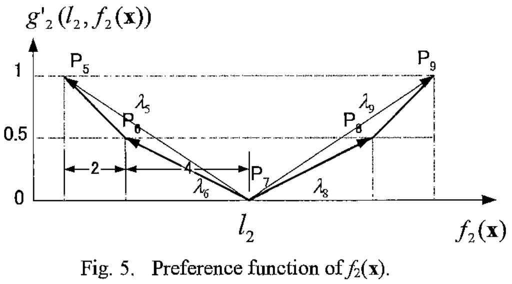

Fig. 5. Preference function of $f_2(\mathbf{x})$.

where n_j, p_j are the negative and the positive deviational variables related to the point P_j, $j = 1, 2, \ldots, 9$.

The minmax GP model for this MOLP problem becomes

$$\min\ \alpha, \quad \cdots\cdots (42)$$

$$\text{subject to } \alpha \geq n_2 + 2p_3, \quad \cdots\cdots (43)$$

$$\alpha \geq \frac{1}{4}n_6 + \frac{1}{8}n_7 + \frac{1}{8}p_7 + \frac{1}{4}p_8, \quad \cdots\cdots (44)$$

Eq. (33)- Eq. (35), Eq. (37)- Eq. (41).

The linear coordination model based on the convex cone concept (LCC) in section 3.2 for this MOLP problem becomes

$$\min\ \lambda_1 + \lambda_4 + \lambda_5 + 0.5\lambda_6 + 0.5\lambda_8 + \lambda_9, \quad \cdots\cdots (45)$$

$$\text{subject to } x_1 + \lambda_1 - 2\lambda_3 - 2.5\lambda_4 = l_1, \quad \cdots\cdots (46)$$

$$\lambda_1 + \lambda_3 + \lambda_4 \leq 1, \quad \cdots\cdots (47)$$

$$x_2 + 6\lambda_5 + 4\lambda_6 - 4\lambda_8 - 6\lambda_9 = l_2, \quad \cdots\cdots (48)$$

$$\lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 \leq 1, \quad \cdots\cdots (49)$$

Eq. (33) - Eq. (35).

The minmax model of the linear coordination method based on the convex cone concept (minmax CC) in section 3.2 for this MOLP problem becomes

$$\min\ \alpha, \quad \cdots\cdots (50)$$

$$\text{subject to } \alpha \geq \lambda_1 + \lambda_4, \quad \cdots\cdots (51)$$

$$\alpha \geq \lambda_5 + 0.5\lambda_6 + 0.5\lambda_8 + \lambda_9, \quad \cdots\cdots (52)$$

Eq. (33)- Eq. (35), Eq. (46)- Eq.(49).

The satisfactory effective linear coordination model based on the convex cone concept (ECC) in Eq. (24) can formulate this MOLP problem as follows.

$$\text{lex min} \begin{bmatrix} \max(\lambda_4, 0.5\lambda_8 + \lambda_9), (\lambda_4 + 0.5\lambda_8 + \lambda_9), \\ \max(-\lambda_1, -\lambda_5 - 0.5\lambda_6), (-\lambda_1 - \lambda_5 - 0.5\lambda_6) \end{bmatrix} \quad \cdots\cdots (53)$$

subject to Eq. (33)- Eq. (35), Eq. (46)- Eq.(49).

The solution results can be shown in Table 1:

Table 1 shows the optimal solutions obtained by different methods solving by linear programming. Note that WGP, LCC minmax GP and minmax CC have generated some of non-efficient solutions. Their results for each aspiration vector are identical. Only one solution in the solution ranges for the 4 aspiration vectors ((2,5),(3,4),(4,4),(5,3)) of the 9 aspiration vectors contains the efficient solutions, while other solutions in these ranges are non-efficient solutions. These methods are based on the concept of satisficing so they try to reach the desired target as much as possible, without considering the efficiency of solutions so some of them may not be efficient solutions. With the proposed method, the satisfactory efficient linear coordination method based on the convex cone concept, a single solution that dominates the GP solution is obtained and this solution is also close to the aspiration levels. All of the solutions obtained by this method are efficient. This is the advantage of this method over existing MOLP methods

Table 1. Optimal solutions from different methods solving by linear programming.

l_1,l_2	WGP & LCC	Minmax GP & minmax CC	Proposed method
1,8	*[2,3],8	*[2,3],8	2,6
2,8	*[2,4],8	*[2,4],8	2.73,5.45
2,5	![3.33,4],5	![3.33,4],5	3.82,4.64
3,4	![4.66,5],4	![4.66,5],4	4.91,3.82
4,4	![4.66,5],4	![4.66,5],4	3.77,4.67
4,5	*[4,6],5	*[4,6],5	3,5.25
5,4	*[5,7],4	*[5,7],4	4,4.5
5,5	*[5,7],5	*[5,7],5	6,3
5,3	![6,7],3	![6,7],3	6,3

* Non-efficient solutions

! The solutions contain efficient solution and non-efficient solutions

for convex polyhedral preference functions.

4.2 A Fuzzy MOLP Problem with Concave Polyhedral Membership Functions In reality, the decision makers may have only vague or imprecise knowledge about trade-off relationships among goals. So applying fuzzy set theory to goals has the advantage of allowing for vague aspirations of decision makers, which can be quantified by some natural language terms [25] [28]. Assuming that goals can be expressed by eliciting the corresponding membership functions. Fuzzy goals are used to define aspiration levels with respect to the linguistic terms, which can be considered as concave polyhedral membership functions, the degree to which these goals are obtained. In this section, the same problem as in section 4.1 is reconsidered using fuzzy preference functions, which are defined in Appendix A.

Given that the decision maker has expressed the following vague aspirations:

$f_1(\mathbf{x})$ is "around [l_1, l_1+2] but rather greater" and its importance is "high".

$f_2(\mathbf{x})$ is "around l_2" and its importance is "low".

Let l_1 and l_2 represent the aspiration or reference levels of objective 1 and objective 2 accordingly and tolerance intervals of objective 1 and objective 2 are

$int_1 = 4$ and $int_2 = 3$.

Then,

Breakpoints of $f_1(\mathbf{x})$ are [(l_1-2,0), (l_1,1),(l_1+2,1),(l_1+3,0)].

Breakpoints of $f_2(\mathbf{x})$ are [(l_2-1.5,0), (l_2,1),(l_1+1.5,0)].

The concave membership functions and their inverse functions for minimization problems can be shown as follows. In this problem the inverse functions represent the convex polyhedral preference functions of objectives.

The additive model for fuzzy goal programming [41] becomes

$$\min \quad 0.5n_2 + p_3 + \frac{2}{3}n_6 + \frac{2}{3}p_6, \quad \cdots (54)$$

$$\text{subject to} \quad x_1 + n_2 - p_2 = l_1, \quad \cdots (55)$$

$$x_1 + n_3 - p_3 = l_1 + 2, \quad \cdots (56)$$

$$x_2 + n_6 - p_6 = l_2, \quad \cdots (57)$$

Eq. (33) - Eq. (35).

where n_j, p_j are the negative and positive deviational variables related to the point P_j, j = 1, 2, …, 7.

The minmax GP model of this problem becomes

$$\min \quad \alpha, \quad \cdots (58)$$

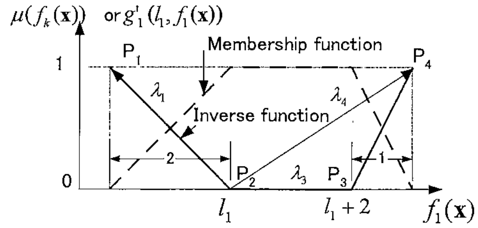

Fig. 6. The preference function of $f_1(\mathbf{x})$.

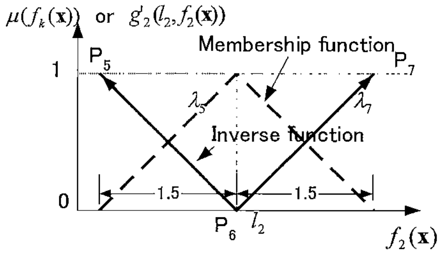

Fig. 7. The preference function of $f_2(\mathbf{x})$.

$$\text{subject to} \quad \alpha \geq 0.5n_2 + p_3, \quad \cdots (59)$$

$$\alpha \geq \frac{2}{3}n_6 + \frac{2}{3}p_6, \quad \cdots (60)$$

Eq. (33) – Eq. (35), Eq. (55) – Eq. (57).

The linear coordination model based on the convex cone concept (LCC) for fuzzy goals becomes

$$\min \quad \lambda_1 + \lambda_4 + \lambda_5 + \lambda_7, \quad \cdots (61)$$

$$\text{subject to} \quad x_1 + 2\lambda_1 - 2\lambda_3 - 3\lambda_4 = l_1, \quad \cdots (62)$$

$$\lambda_1 + \lambda_3 + \lambda_4 \leq 1, \quad \cdots (63)$$

$$x_2 + 1.5\lambda_5 - 1.5\lambda_7 = l_2, \quad \cdots (64)$$

$$\lambda_5 + \lambda_7 \leq 1, \quad \cdots (65)$$

Eq. (33) – Eq. (35), $\lambda_j \geq 0$, j = 1, 2, …, 7.

where λ_j is the coefficient related to breakpoint P_j, j = 1,2,…,7.

The minmax model of the linear coordination based on the convex cone concept (minmax CC) for fuzzy goals becomes

$$\min \quad \alpha, \quad \cdots (66)$$

$$\text{subject to} \quad \alpha \geq \lambda_1 + \lambda_4, \quad \cdots (67)$$

$$\alpha \geq \lambda_5 + \lambda_7, \quad \cdots (68)$$

Eq. (33) – Eq. (35), Eq. (62)- Eq. (65), $\lambda_j \geq 0$, j = 1, 2, …, 7.

The satisfactory effective linear coordination model based on the convex cone concept (ECC) can be formulated fuzzy goals as

$$\text{lex min}\left[\max_{1\leq k\leq 2}(\lambda_4, \lambda_7), (\lambda_4 + \lambda_7), \max_{1\leq k\leq 2}(-\lambda_1, -\lambda_5), (-\lambda_1 - \lambda_5)\right] \quad \cdots (69)$$

subject to Eq. (33) – Eq. (35), Eq. (62)- Eq. (65),

$\lambda_j \geq 0$, j = 1,2,…,7.

Table 2. Optimal solutions from different methods solving by linear programming.

l_1, l_2	Additive model & LCC	Minmax GP & minmax CC	Proposed method
1,8	*[2,3],8	*[2,3],8	2,6.5
2,8	*[2,4],8	*[2,4],8	2,6.5
2,5	![3.33,4],5	![3.33,4],5	4,4.5
3,4	![4.66,5],4	![4.66,5],4	5,3.75
4,4	![4.66,6],4	![4.66,6],4	6,3
4,5	*[4,6],5	*[4,6],5	5.33,3.5
5,4	*[5,7],4	*[5,7],4	6,3
5,5	*[5,7],5	*[5,7],5	5.33,3.5
5,3	![6,7],3	![6,7],3	6,3

* Non-efficient solution

! The solutions contain efficient solution and non-efficient solutions

The solutions result comparing with existing methods can be shown in Table 2.

Table 2 shows that the optimal solutions obtained by the proposed method can provide all of the efficient solutions, which are also close to the aspiration levels. Note that the additive model of fuzzy goals, LCC, minmax GP and minmax CC have generated some non-efficient solutions. Only one solution in the solution ranges for 4 aspiration vectors ((2,5), (3,4), (4,4), (5,3)) of the 9 aspiration vectors contains the efficient solutions, while other solutions in these ranges are non-efficient solutions. These methods are based on the concept of satisficing so they try to reach the desired target as much as possible without considering the efficiency of solutions. With the proposed method, the satisfactory efficient linear coordination method based on the convex cone concept, a single solution that dominates the minmax GP solution is obtained and this solution is also close to the aspiration levels. All of the solutions obtained by this method are efficient. This is the advantage of this method over existing fuzzy MOLP methods for concave polyhedral membership functions.

5. Conclusion

The satisfactory efficient linear coordination method for convex polyhedral preference functions of an MOLP problem is proposed in this research. The existing efficient methods take preference functions as a certain target point preference, which is too rigid. Convex polyhedral preference functions are preferable. The concept of convex cone and the existing lexicographic model of RPM are applied to the efficient linear coordination method, which can be easily formulated and solved by linear programming. The solution obtained by this method is always efficient and also close to the decision maker's requirements. This method also provides a better solution than the existing methods in terms of Pareto optimality. Furthermore, the flexibility in designing preference functions is also enhanced.

For imprecise situations, a flexible membership function, the convex polyhedral function, can be used to represent fuzzy MOLP membership functions stated by the decision maker. The satisfactory efficient linear coordination method developed here can be applied to this fuzzy MOLP problem. A satisfactory efficient solution can be found by this method, which is also better than the existing fuzzy MOLP methods for concave polyhedral membership functions.

The proposed method can use linear formulation to solve nonlinear preference functions in the case of both fuzzy and non-fuzzy MOLP problems. Moreover, it can generate a single satisfactory efficient solution for the decision maker with only one reference point. In the future, we hope to develop this method as an interactive approach. Also, we would like to do more work on the application of these methods to real world situations.

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Appendix A

For each objective of a fuzzy MOLP problem, the following linguistic term can be used.

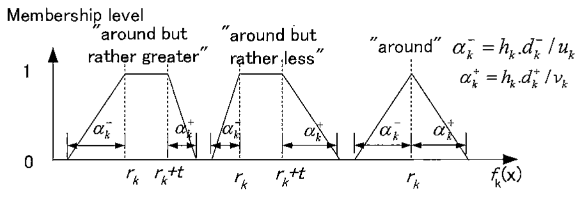
With "around", "around but rather greater" and "around but rather less" situations, α_k^+ and α_k^- are the respective spreads to right and left of r_k, where r_k can be a single reference membership value or an interval reference membership value denoted by $[r_k, r_k+t_k]$. Under "at most" situation, we allow the kth goal to be spread to the right hand side of r_k with a certain range β_k^+. Similarly, with "at least" β_k^- is the allowed left spread of r_k, as shown in app. Fig. 1 and app. Fig. 2.

Membership functions of an aspiration level, $\mu_k(f_k(\mathbf{x}))$ of $f_k(\mathbf{x})$ with respect to three linguistic types can be expressed as:
(Type-1: "around", "around but rather greater", "around but rather less"):

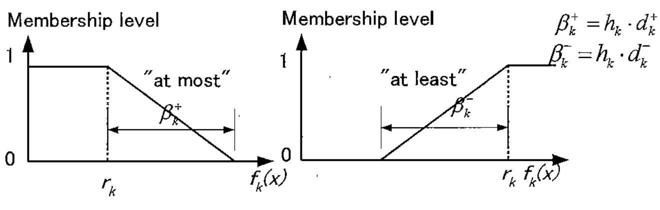
$$\mu_k(f_k(\mathbf{x}))=\begin{cases}0, & \text{if } f_k(\mathbf{x})\le r_k-h_k\cdot d_k^-/u_k,\\ \left(1-\dfrac{[r_k-f_k(\mathbf{x})]}{h_k\cdot d_k^-}u_k\right), & \text{if } r_k-h_k\cdot d_k^-/u_k\le f_k(\mathbf{x})\le r_k,\\ 1, & \text{if } r_k\le f_k(\mathbf{x})\le r_k+t_k,\\ \left(1-\dfrac{[f_k(\mathbf{x})-(r_k+t_k)]}{h_k\cdot d_k^+}v_k\right), & \text{if } r_k+t_k\le f_k(\mathbf{x})\le r_k+t_k+h_k\cdot d_k^+/v_k,\\ 0, & \text{if } f_k(\mathbf{x})\ge r_k+t_k+h_k\cdot d_k^+/v_k,\end{cases} \quad \cdots\cdots \text{(A1)}$$

where u_k, $v_k \in \{0.5,1\}$, $k = 1,2,\ldots,K$,
d_k^+, d_k^- are tolerance intervals on the negative and positive side of r_k.

If $t_k = 0$, then the trapezoidal membership function is reduced to the triangular membership function as shown in app. Fig. 1. u_k and v_k are equal to 1 for "around", u_k is equal to 0.5 and v_k is equal to 1 for "around but rather greater" and u_k is equal to 1 and v_k is equal to 0.5 for "around but rather less", respectively. We allow the membership function of type-1 goals to have a different shape of deviations simultaneously. h_k are the constant values that present the decision maker attitude towards the objective function's importance ("low", "medium", "high"). h_k are assigned to be 1 for "low" and 0.75 for "medium" and 0.5 for "high", respectively. A lower deviation from the desired target



app. Fig. 1. The membership function of type-1.



app. Fig. 2. The membership functions of type-2 and type-3.

means a higher level of important of that objective.

(Type-2: "at most"):

$$\mu_k(f_k(\mathbf{x})) = \begin{cases} 1, & \text{if } f_k(\mathbf{x}) \le r_k, \\ \left(1 - \dfrac{[f_k(\mathbf{x}) - r_k]}{h_k \cdot d_k^+}\right), & \text{if } r_k \le f_k(\mathbf{x}) \le r_k + h_k \cdot d_k^+, \\ 0, & \text{if } f_k(\mathbf{x}) \ge r_k + d_k^+, \end{cases} \quad \cdots\cdots \text{(A2)}$$

(Type-3: "at least"):

$$\mu_k(f_k(\mathbf{x})) = \begin{cases} 0, & \text{if } f_k(\mathbf{x}) \le r_k - h_k \cdot d_k^-, \\ \left(1 - \dfrac{[r_k - f_k(\mathbf{x})]}{h_k \cdot d_k^-}\right), & \text{if } r_k - h_k \cdot d_k^- \le f_k(\mathbf{x}) \le r_k, \\ 1, & \text{if } r_k \le f_k(\mathbf{x}), \end{cases} \quad \cdots\cdots \text{(A3)}$$

Breakpoints of each type of membership function in 2-dimensional space can be summarized as:

Type-1, B_{kd} are

$$\left[(r_k - h_k \cdot d_k^- / u_k, 0), (r_k, 1), (r_k + t_k, 1), (r_k + t + h_k \cdot d_k^- / v_k, 0)\right],$$

for an interval aspiration level and if r_k is a single value the third element is not included.

Type-2 and Type-3, B_{kd} are

$$\left[(r_k, 1), (r_k + h_k \cdot d_k^+, 0)\right] \text{ and } \left[(r_k - h_k \cdot d_k^-, 0), (r_k, 1)\right]$$

d_k^+ and d_k^- can be desired by decision makers or we can desire by tolerance interval of Zimmermann's approach (32)(33)(35)(38). With Zimmermann's approach, we first calculate the individual minimum of each objective function $f_k(\mathbf{x})$ under given constraints. From this calculation, the values of other objective functions also can be found. By taking account of the calculated values, individual minimum and maximum of each objective function are selected to be $f_k(\mathbf{x})^{\max}$ and $f_k(\mathbf{x})_{\min}$. Then, the tolerance interval can be calculated as, $int_k = f_k(\mathbf{x})^{\max} - f_k(\mathbf{x})_{\min}$.

In our case, we assume that the tolerance interval from Zimmermann's approach is assigned for each type of membership function as follows:

Type-1 assume $d_k^+ = d_k^- = int_k/2$,

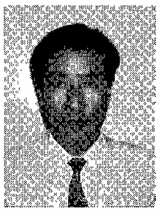
Type-2 assume $d_k^+ = int_k$ and,

Type-3 assume $d_k^- = int_k$,

where int_k is the tolerance interval of kth objective, $k = 1,2,\ldots,K$.

Aspiration levels with respect to the linguistic terms can be used to define the concave polyhedral membership functions. These goal membership functions represent the degrees of satisfaction at different objective levels. From the primitive minimization problem of MOLP with convex polyhedral preference functions, Eq. (24) is formulated. The convex preference functions of this problem are considered as the inverse functions of membership functions because the highest satisfaction level corresponds to the lowest level of the objective value.

Busaba Phruksaphanrat (Non-member) received the BS degree in Industrial Engineering from Thammasat University, Thailand in 1994, MS degree in Manufacturing System Engineering from Asian Institute of Technology, Thailand. Presently, she is a doctoral student of Graduate School of Engineering, Nagaoka University of Technology. Her research interests include linear and nonlinear optimization, multi-objective optimization, fuzzy system and applications of manufacturing systems.



Ario Ohsato (member) received the D.Eng. degree from Department of System Science, the Tokyo Institute of Technology in 1987. Currently, he is a professor of Department of Management and Information Systems Science, Nagaoka University of Technology. His research areas are system engineering, fuzzy engineering, optimization and their applications. He is a member of IEEJ, SICE and IFSA.